

Lecture 1

• <https://www.ibs.re.kr/ecopro/course-discrete-geometry/>

• Search ECOPRO ibs → Events → online course

1. Matoušek : Lectures on Discrete Geometry

2. Barany : Combinatorial convexity.

Plan

- Basic notions :
 - affine
 - convex
 - sep. thm / Farkas's Lem.

- 3 basic properties of convexity in $\mathbb{R}^d$

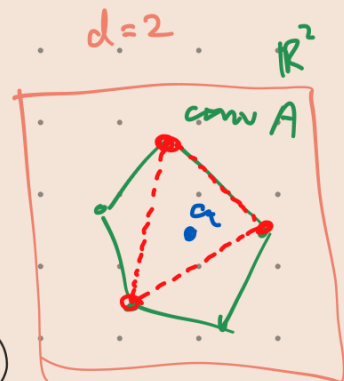
Carathéodory

$$\forall A \subseteq \mathbb{R}^d$$

$$\forall a \in \text{conv } A \Rightarrow \exists A' \subseteq A : |A'| \leq d+1 \text{ s.t. } a \in \text{conv } A'$$

Radon

Helly



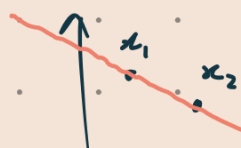
§ affine notion in $\mathbb{R}^d$

- affine subspace : $x + L$ where $L \subseteq \mathbb{R}^d$ is a linear subsp.

- affine combination of $\{x_1, \dots, x_n\} = X$

$$\left\{ \sum_{i=1}^n \alpha_i x_i : \sum \alpha_i = 1 \right\} = \text{aff } X$$

↑ affine hull of X



One way to think of it:

- translate by $-x_n$: $x_1 - x_n, \dots, x_{n-1} - x_n$
- take all linear comb: $\sum_{i=1}^{n-1} \beta_i (x_i - x_n)$ + x_n
- translate back: →

- affine dependence: if $x_n \in \text{aff} \{x_1, \dots, x_{n-1}\}$, then $x_1, \dots, x_n$ are aff. dependent.

$$\sum_{i=1}^{n-1} \alpha_i x_i = x_n$$

$$\sum \alpha_i = 1$$

$\Leftrightarrow$

$\exists \beta_i$ not all 0 s.t.

$$\sum_{i=1}^n \beta_i x_i = 0$$

$$\sum_{i=1}^n \beta_i = 0$$

$$\sum_{i=1}^n \beta_i \begin{pmatrix} x_i \\ 1 \end{pmatrix} = 0$$

(*) aff. dep. $x_1, \dots, x_n \Leftrightarrow x_1 - x_n, \dots, x_{n-1} - x_n$ linear dep.

$\Leftrightarrow \begin{pmatrix} x_i \\ 1 \end{pmatrix}$ linear dep. in $\mathbb{R}^{d+1}$

(*) max # aff. indep. pts in $\mathbb{R}^d = d+1$

• natural correspondence between aff. notion in $\mathbb{R}^d \leftrightarrow$ linear notion in $\mathbb{R}^{d+1}$

Embed $\mathbb{R}^d \rightarrow \mathbb{R}^{d+1}$ as $x_{d+1} = 1$

aff. subsp. $\leftrightarrow$ linear subsp.

dim k

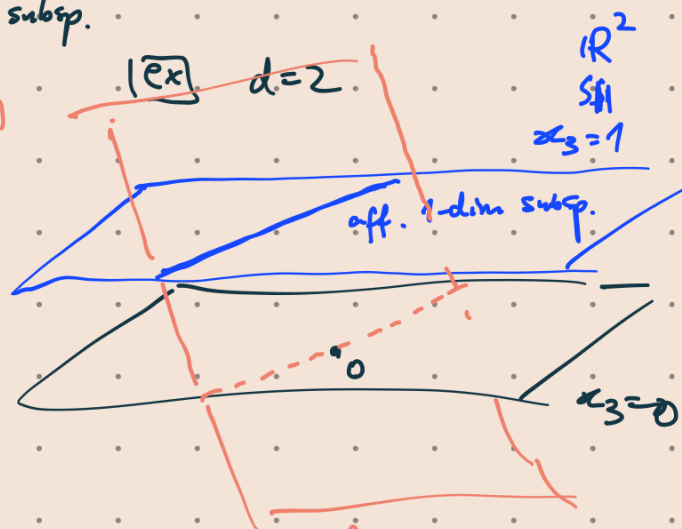
dim $(k+1)$

$\mathbb{R}^d$

• k dim aff. subsp.: k -flat

- $k = d-1$: hyperplane $\sum_{i=1}^d \alpha_i x_i = b$
 $\alpha \cdot x$

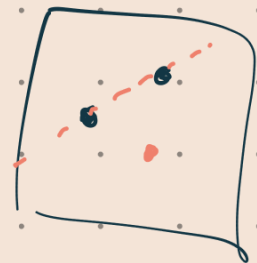
- $k = 1$: line



• General position: "no unlikely configuration"

No $k+2$ pts lie in one k -flat

$\rightarrow k=1$



$\mathbb{R}^2$

§ convex sets 'no dent'



Def: $C \subseteq \mathbb{R}^d$ is convex if

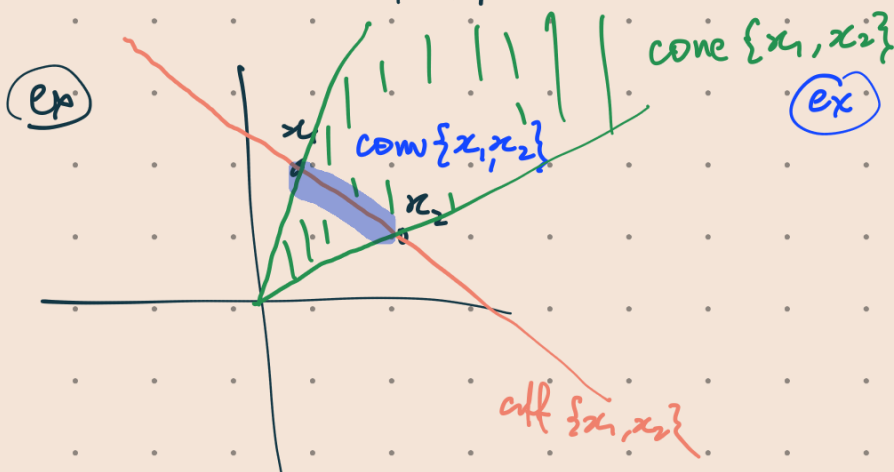
$\forall x, y \in C$, seg. xy in C .

i.e. $\forall t \in [0, 1]$, $tx + (1-t)y \in C$.

$\mathbb{R}^d$ closed under intersection

Def: Convex hull of $X = \{x_1, \dots, x_n\}$

$$\text{conv } X = \left\{ \sum_{i=1}^n t_i x_i : \sum t_i = 1, t_i \geq 0 \forall i \right\}$$



Def cone of $X = \{x_1, \dots, x_n\}$

$$\text{cone } X = \left\{ \sum_{i=1}^n \alpha_i x_i : \alpha_i \geq 0 \right\}$$

§ Separation theorem.



Thm $\forall$ disjoint convex set $C, D \in \mathbb{R}^d$

$\Rightarrow \exists$ hyperplane h separating them.

i.e. $\exists a \in \mathbb{R}^d : \text{s.t. } a \cdot x \geq b \quad \forall x \in C$
 $\leq b \quad \forall x \in D$

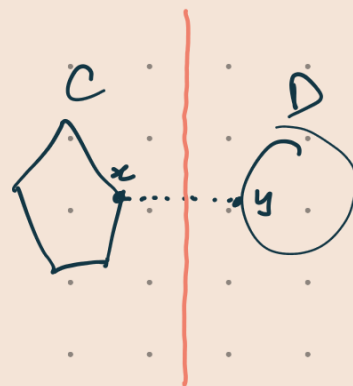
If C, D are closed & ≥ 1 bounded $\Rightarrow$ strict sep.

$$h \cap C = h \cap D = \emptyset$$

Pf [Sketch] : both C & D compact (bdd & closed)

$\Rightarrow$ dist. funct. $\|x - y\|$ attains
its min. on $C \times D$

• Take a hyperplane $\perp$ seg xy . 



Lem (Farkas lem) $\forall A \in \mathbb{R}^{d \times n} \quad \forall b \in \mathbb{R}^d$. Then exactly one of the following holds.

1. $\exists x \in \mathbb{R}^n \quad \text{s.t.} \quad Ax = b$
 $x \geq 0$

2. $\exists y \in \mathbb{R}^d \quad \text{s.t.} \quad \left. \begin{array}{l} A^T y \geq 0 \\ b^T y < 0 \end{array} \right\}$

Rmk : When there is no solⁿ as in 1.?

Obvious obstruction : linear sys. is inconsistent $\Rightarrow -1 = 0$

$$0 \leq \underline{\underline{y^T A x}} = y^T b < 0$$

So Farkas lem $\Rightarrow$ such obvious obstruction is the

ONLY obstruction to having a solⁿ

Farkas 1. $\exists x \in \mathbb{R}^n \quad \text{s.t.} \quad Ax = b$

$$\left\{ \begin{array}{l} \text{OR} \\ 2. \exists y \in \mathbb{R}^d \text{ s.t.} \end{array} \right. \quad x \geq 0$$

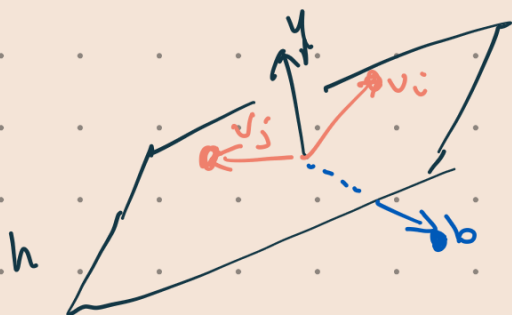
$$\left. \begin{array}{l} A^T y \geq 0 \\ b^T y < 0 \end{array} \right\}$$

$$A = \begin{pmatrix} | & | & | & | & | \\ u_1 & & & & u_n \\ | & | & | & | & | \end{pmatrix}_{d \times n} \quad u_i \in \mathbb{R}^d$$

Pf [Sketch] Consider cone $\{u_1, \dots, u_n\} = \{Ax : x \geq 0\}$

1st alternative $\Rightarrow \exists \text{ sol}^n \Leftrightarrow b \in \text{cone } A$

2nd alternative $\Rightarrow y$ normal vector of a hyperplane



If 1. does not hold
 $\Rightarrow$ sep. thm gives a hyperplane h

$\rightarrow y$ is 2nd alternative.

Farkas Lem. var.

• either $\exists x$ s.t. $Ax \leq b$

• or $\exists y \geq 0$ s.t. $A^T y = 0$
 $b^T y < 0$

Farkas lem. var. (Geometric way)

$\forall a_1, \dots, a_n \in \mathbb{R}^d \quad \forall b \in \mathbb{R}^n$

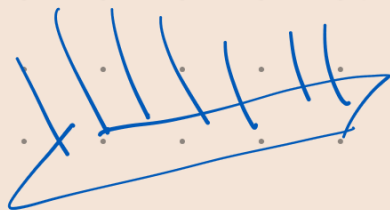
$$a_1 \cdot x \leq b_1$$

$$\vdots$$

$$a_n \cdot x \leq b_n$$

$$\Leftrightarrow Ax \leq b$$

$$A = \begin{pmatrix} -a_1 \\ \vdots \\ -a_n \end{pmatrix}$$



These closed half-spaces

have no common pts

(i.e. $Ax \leq b$ no solⁿ)

$$\Leftrightarrow$$

$$\begin{pmatrix} 0 \\ -1 \end{pmatrix} \in \text{cone} \left\{ \begin{pmatrix} a_i \\ b_i \end{pmatrix} \in \mathbb{R}^{d+1} : i \in [n] \right\}$$

