Lecture 9. Extremal set theory

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We now turn to another central area in extremal combinatorics, which studies set systems (or hypergraphs). We will sample a couple of beautiful results in this area, and see 1) how results here connect to other areas, e.g. geometry, and 2) how tools from e.g. probability and linear algebra can be used to prove statements here. The themes of the results we shall see are nicely captured by the following quote of Tibor.

What do humans do? Well, they think and they love. What do sets do? They intersect and they contain. –Tibor Szabó.

1 Sperner's theorem

A *partial order set*, or *poset*, $(P, \prec)$ is a set P with a binary relation $\prec$ that is reflexive ($\forall x, x \prec x$), antisymmetric ($x \prec y, y \prec x \Rightarrow x = y$) and transitive ($x \prec y, y \prec z \Rightarrow x \prec z$). Two elements x, y are *comparable* if $x \prec y$ or $y \prec x$. A subset $C \subseteq P$ is a *chain* if elements in C are pairwise comparable; and a subset $A \subseteq P$ is called an *antichain* if elements in A are pairwise incomparable.

For example, the *Boolean poset* $(2^{[n]}, \subseteq)$ is the poset on the family of all subsets of $[n]$ with containment relation. A family of sets $\mathcal{F}$ is a chain if non-containment is forbidden, i.e. sets in $\mathcal{F}$ can be ordered as $F_1, \dots, F_i, \dots$ such that for any $i < j$, $F_i \subseteq F_j$. Here, antichains forbid containment, i.e. no set is a subset of another.

The classical Sperner's theorem states that in a Boolean poset, the size of the largest antichain is the same as that of the middle layer.

Theorem 1.1 (Sperner 1928). *If $\mathcal{F} \subseteq 2^{[n]}$ is an antichain, then $|\mathcal{F}| \le \binom{n}{\lfloor n/2 \rfloor}$.*

Sperner's theorem follows immediately from the Lubell–Yamamoto–Meshalkin inequality.

Theorem 1.2 (LYM inequality). *For any antichain $\mathcal{F} \subseteq 2^{[n]}$, $\sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}} \le 1$.*

We will leave as an exercise to derive LYM inequality from Bollobás set-pairs inequality which will be introduced shortly.

2 Bollobás set-pairs inequality

Theorem 2.1 (Bollobás 65). *Given sets $A_1, \dots, A_m, B_1, \dots, B_m$, if*

- *$A_i \cap B_i = \varnothing$ for all $i \in [m]$; and*
- *$A_i \cap B_j \neq \varnothing$ for any $i \neq j$,*

then

$$\sum_{i \in [m]} \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}} \le 1.$$

In particular, if for all $i \in [m]$, $|A_i| = a$, $|B_i| = b$, then $m \le \binom{a+b}{a}$.

Proof. Let $\Omega = \cup_{i\in[m]}(A_i \cup B_i)$ be the ground set. Without loss of generality, say $\Omega = [n]$.

Take a permutation $\sigma \in S_n$ uniformally at random over all permutations on $[n]$. For each $i \in [m]$, let P_i denote the event that A_i precedes B_i in σ, that is, $\max \sigma(A_i) < \min \sigma(B_i)$. Note that the event P_i depends only on relative positions of $\sigma(A_i)$ and $\sigma(B_i)$ in $\sigma(A_i \cup B_i)$. Since there are exactly $|A_i|! \cdot |B_i|!$ many ways to arrange all elements of A_i to come before all of those in B_i, and σ is a chosen uniformly over S_n, we see that

$$\mathsf{Pr}(P_i) = \frac{|A_i|! \cdot |B_i|!}{(|A_i| + |B_i|)!}.$$

It suffices then to show that P_i, $i \in [m]$, are pairwise disjoint events. Indeed, this would imply

$$\sum_{i\in[m]} \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}} = \sum_{i\in[m]} \frac{|A_i|! \cdot |B_i|!}{(|A_i| + |B_i|)!} = \sum_{i\in[m]} \mathsf{Pr}(P_i) = \mathsf{Pr}\big(\cup_{i\in[m]} P_i\big) \le 1.$$

Take distinct $i, j \in [m]$ and suppose that P_i occurs, i.e. $\max \sigma(A_i) < \min \sigma(B_i)$. As both $A_i \cap B_j$ and $A_j \cap B_i$ are non-empty, we get that

$$\min \sigma(B_j) \le \max \sigma(A_i) < \min \sigma(B_i) \le \max \sigma(A_j).$$

Thus, P_j does not occur as desired. □

Exercise 2.2. Give an example showing that Theorem 2.1 is tight.

2.1 Skewed version

Theorem 2.1 was generalised independently by Lovász, Alon and Kalai to a skewed version as follows.

Theorem 2.3 (Lovász 70s)**.** *Given sets $A_1, \dots, A_m, B_1, \dots, B_m$, if*

- $|A_i| \le a$, $|B_i| \le b$ *for all* $i \in [m]$*; and*
- $A_i \cap B_i = \varnothing$ *for all* $i \in [m]$*; and*
- $A_i \cap B_j \neq \varnothing$ *for any* $j > i$*,*

then

$$m \le \binom{a+b}{a}.$$

Remark 2.4. While there is a short combinatorial proof of Bollobás set-pairs inequality, for the skewed version, Theorem 2.3, only a linear algebraic proof is known. It would be interesting to have a combinatorial one.

We will see a proof of the skewed version using exterior algebra. Let us first collect the useful basics. The exterior algebra $\bigwedge V$ of a vector space V over a field K is defined as the quotient algebra of the tensor algebra by the two-sided ideal I generated by all elements of the form $x \otimes x$, $x \in V$. For any $x \in V$, we have $x \wedge x = 0$. The k-th exterior power of V, denoted by $\bigwedge^k V$, is the vector subspace of $\bigwedge V$ spanned by elements of the form

$$x_1 \wedge x_2 \wedge \ldots \wedge x_k,\ x_i \in V, i \in [k].$$

In particular, $x_1 \wedge x_2 \wedge \ldots \wedge x_k = 0$ if $x_i = x_j$ for some distinct $i, j \in [k]$.

If the dimension of V is n, then

$$\dim \bigwedge^k V = \binom{n}{k}.$$

Proof of Theorem 2.3. We may assume that each A_i has size exactly a. Assume that $[n] = \cup_{i\in[m]}(A_i \cup B_i)$ is the ground set. Let $V = \mathbb{R}^{a+b}$ be the $(a+b)$-dimension Euclidean space, and let $v_1, \ldots, v_n$ be vectors in general position in V, i.e. every set of at most $a+b$ of them is linearly independent.

For each $i \in [m]$, let $y_i = \bigwedge_{j\in A_i} v_j \in \bigwedge^a V$ and $z_i = \bigwedge_{k\in B_i} v_k$. Then as v_is are in general position, we get from $A_i \cap B_i = \varnothing$ that

$$y_i \wedge z_i \neq 0; \tag{1}$$

on the other hand, for any $j > i$, $A_i \cap B_j \neq \varnothing$ implies that

$$y_i \wedge z_j = 0. \tag{2}$$

To finish the proof, it suffices to show that $y_1, \ldots, y_m$ are linearly independent in $\bigwedge^a V$, since then $m \leq \dim \bigwedge^a V = \binom{a+b}{a}$. Suppose $\alpha_1 y_1 + \ldots + \alpha_m y_m = 0$. Then, by (2), we see that

$$0 = 0 \wedge z_m = (\alpha_1 y_1 + \ldots + \alpha_m y_m) \wedge z_m = \alpha_m y_m \wedge z_m.$$

Thus, due to (1), $\alpha_m = 0$. Repeating this shows that all $\alpha_{m-1}, \ldots, \alpha_1$ are zero as desired. □

3 Applications

Here we give some applications of Sperner's theorem and set-pairs inequalities.

3.1 Littlewood-Offord problem

A classical result of Littlewood-Offord bounds the atom probability of Rademacher sum. It plays an important role in random matrix theory. For instance, it is used to bound the singularity probability of random matrices. Here we present a beautiful proof due to Erdős using Sperner's theorem. Recall that a Rademacher random variable takes value in $\{-1, 1\}$ uniformly.

Theorem 3.1 (Erdős 1945)**.** *Let $a_1, \ldots, a_n \in \mathbb{Z} \setminus \{0\}$ be non-zero integers, and $x_1, \ldots, x_n$ be i.i.d. Rademacher random variables. Then*

$$\sup_{z\in\mathbb{Z}} \mathsf{Pr}\Big(\sum_{i\in[n]} a_i x_i = z\Big) \leq \frac{\binom{n}{\lfloor n/2 \rfloor}}{2^n}.$$

Proof. There are 2^n choices for values of $x_1, \ldots, x_n$. Fix $z \in \mathbb{Z}$ and let $S \subseteq \{-1, 1\}^n$ be the set of choices that results in $\sum_{i\in[n]} a_i x_i = z$. It suffices to show that $|S| \leq \binom{n}{\lfloor n/2 \rfloor}$. By symmetry of Rademacher random variables, we may assume all $a_1, \ldots, a_n$ are positive.

Identify S with a subset of Boolean poset as follows: each vector $(y_1, \ldots, y_n) \in S$ corresponds to a set $Y \subseteq [n]$ where each $i \in [n]$ falls in Y if and only if $y_i = 1$. Now, if S contains two comparable elements, say $Y \subseteq Y'$ or $(y_1, \ldots, y_n) \prec (y'_1, \ldots, y'_n)$, then as all a_is are positive, we get that

$$0 = \sum_{i\in[n]} a_i y'_i - \sum_{i\in[n]} a_i y_i = 2 \sum_{i\in Y'\setminus Y} a_i > 0,$$

a contradiction.

Thus, S is an antichain in $2^{[n]}$ and by Sperner's theorem $|S| \leq \binom{n}{\lfloor n/2 \rfloor}$ as desired. □

We remark that the bound here is optimal. Indeed, consider the case when n is even, all $a_i = a$ are equal and $z = 0$. Then the probability that $\sum_{i\in[n]} a x_i = 0$ is precisely $\binom{n}{n/2}/2^n$ as we have to choose precisely half of the elements to be 1.

3.2 LYM inequality

The first one is that it implies the LYM inequality and Sperner's theorem.

Exercise 3.2. Prove the LYM inequality, Theorem 1.2, using Theorem 2.1.

3.3 Saturation number

Given a k-graph (k-uniform hypergraph) H, we say a k-graph G is *H-saturated* if G is H-free, but adding any edge to G would create a copy of H, i.e. $H \subseteq G \cup \{e\}$ for any $e \in \binom{V(G)}{k} \setminus E(G)$. In other words, a graph is H-saturated if it is maximally H-free. The *saturation number* of H is defined as follows:

$$\mathrm{sat}(n, H) = \min\{e(G) : \ G \text{ is an } n\text{-vertex } H\text{-saturated graph}\}.$$

For example, take $H = K_3$ to be the triangle, then clearly the bipartite Turán graph $K_{n/2,n/2}$ is K_3-saturated, which has quadratic number of edges. By definition, we always have $\mathrm{sat}(n, H) \leq \mathrm{ex}(n, H)$. But how small can the saturation number be? Note that the star $K_{1,n-1}$ is also K_3-saturated and has merely $n - 1$ edges. Can we have even fewer edges? We shall see below that the answer is no. While the Turán problem for hypergraphs are wide open, we can resolve the saturation problem completely. We write $K_t^{(k)}$ for the complete k-graph on t vertices.

Exercise 3.3 (Bollobás 65)**.** Let $n \geq t \geq k \geq 2$. Then

$$\mathrm{sat}(n, K_t^{(k)}) = \binom{n}{k} - \binom{n-t+k}{k}.$$