Lecture 7. $(6, 3)$-theorem, Roth's theorem

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In this lecture, we will see some applications of triangle removal lemma.

1 $(6, 3)$-theorem

In this section, we present Ruzsa-Szemerédi $(6, 3)$-theorem, which implies Roth's theorem on 3-term arithmetic progression (3AP). This can be viewed as a Turán type problem for hypergraphs.

Throughout this section, we will work with 3-uniform hypergraphs $\mathcal{H} = (V, E)$, where the edge set $E \subseteq \binom{V}{3}$ consists of triples in V. We say a hypergraph is *linear* if any two of its edges share at most one vertex in common. For $s, t \in \mathbb{N}$, an (s, t)-configuration (or simply (s, t)) in a hypergraph is a set of s vertices inducing at least t edges. A hypergraph is (s, t)*-free* if it does not contain any (s, t)-configurations.

Theorem 1.1 ((6,3)-theorem)**.** *If an n-vertex 3-uniform hypergraph $\mathcal{H}$ is $(6, 3)$-free, then*

$$e(\mathcal{H}) = o(n^2).$$

We remark that this upper bound is not very far from optimal: there exists n-vertex 3-uniform $\mathcal{H}$ that is $(6, 3)$-free and have $e(\mathcal{H}) > n^2 \cdot e^{-c\sqrt{\ln n}}$, which is larger than $n^{2-\varepsilon}$ for any constant $\varepsilon > 0$. We shall give this lower bound construction later.

Proof of $(6, 3)$-theorem. Suppose to the contrary that there exists $c > 0$ such that for infinitely many n, there is a $(6, 3)$-free 3-uniform n-vertex $\mathcal{H}$ with $e(\mathcal{H}) > cn^2$. By zooming into a subgraph with higher average degree (which is still a counterexample), we may in addition assume that $\mathcal{H}$ is maximal in the sense that no subgraph of $\mathcal{H}$ has larger average degree than $\mathcal{H}$.

The maximality of $\mathcal{H}$ implies that it is linear. Indeed, if there are two edges intersecting at two points, we have a $(4, 2)$-configuration, then no other edges intersect at these four points, as otherwise we get a $(6, 3)$. Thus these two edges form a component themselves. Then deleting this component results in a subgraph with higher average degree than $\mathcal{H}$, contradicting the maximality of $\mathcal{H}$. Note that, since $\mathcal{H}$ is linear, $\mathcal{H}$ is a partial steiner triple system on n vertices, which is known to have at most $(1/6 + o(1))n^2$ hyperedges.

Let G be the *shadow graph* of $\mathcal{H}$, obtained by setting $V(G) = V(\mathcal{H})$ and turning every hyperedge in $\mathcal{H}$ into a triangle in G. We say a triangle in G is an $\mathcal{H}$*-triangle* if it corresponds to a hyperedge in $\mathcal{H}$. Since $\mathcal{H}$ is linear, no two $\mathcal{H}$-triangles in G share an edge. Thus, there are at least $e(\mathcal{H}) > cn^2$ edge-disjoint $\mathcal{H}$-triangles in G. Consequently, G cannot be made triangle-free by removing at most cn^2 edges, and so the triangle removal lemma implies that G contains at least an^3 triangles in G, where $a = a(c)$. For large n, $an^3 > n^2 > e(\mathcal{H})$, meaning that there are triangles in G that does not come from a hyperedge in $\mathcal{H}$. Such a triangle in G corresponds to a $(6, 3)$ in $\mathcal{H}$ as $\mathcal{H}$ is linear, a contradiction. □

2 Roth's theorem

In this section, we will see a surprising connection between hypergraph Turán type results and number theoretic results. In particular, we will show how $(6,3)$-theorem implies Roth's theorem on 3APs.

Theorem 2.1 (Roth's theorem)**.** *For any $\delta > 0$, there exists n_0 such that for $n \geq n_0$, any subset $S \subseteq [n]$ with size δn contains a three-term arithmetic progression.*

Theorem 2.2. *$(6,3)$-theorem $\implies$ Roth's theorem.*

Proof. Suppose there is a 3AP-free set $A \subseteq [n]$ with $|A| \geq \delta n$. Define a 3-partite 3-uniform hypergraph $\mathcal{H}$ as follows: $V(\mathcal{H}) = V_1 \cup V_2 \cup V_3$, where $V_1 = [n]$, $V_2 = [2n]$ and $V_3 = [3n]$; for the edge set, for each $x \in [n]$ and $a \in A$, add the hyperedge $(x, x+a, x+2a)$. So

$$e(\mathcal{H}) = |A||V_1| \geq \delta n^2.$$

Thus, Theorem 1.1 implies that there is a $(6,3)$-configuration in $\mathcal{H}$. Say the three hyperedges in this $(6,3)$-configuration are $(x, x+a, x+2a)$, $(y, y+b, y+2b)$ and $(z, z+c, z+2c)$. Since two points completely determine an edge in $\mathcal{H}$, this $(6,3)$-configuration has to have two points from each V_i, $1 \leq i \leq 3$. Without loss of generality, say $x = z \neq y$, then $x + a \neq z + c$ and $x + 2a \neq z + 2c$, otherwise two edges coinside. Again without loss of generality, say $y+2b = x+2a$, then similarly $y + b \neq x + a$ and so $y + b = z + c$. Then a simple calculation shows that $b + c = 2a$. Note that $a, b, c \in A$ and $a \neq c$ since $x = z$ and $x + a \neq z + c$. Thus $\{b, a, c\} \subseteq A$ forms a 3AP, a contradiction. □

Dense (6,3)-free $\mathcal{H}$. In the above proof, in fact A is 3AP-free if and only if $\mathcal{H}$ is (6,3)-free. Behrend constructed a 3AP-free subset of $[n]$ of size $n \cdot e^{-c\sqrt{\log n}}$. The corresponding hypergraph $\mathcal{H}$ then is (6,3)-free and has $n^2 \cdot e^{-c\sqrt{\log n}}$ edges.

3 Induced matchings

A matching M in a graph G is an *induced matching* if between any pair of matching edges, there is no other edges, i.e. $E(G[V(M)]) = E(M)$.

Theorem 3.1 (Induced matching theorem)**.** *If an n-vertex graph G is a union of n induced matchings, then $e(G) = o(n^2)$.*

Before we prove it, we give some of its applications: induced matching theorem implies both $(6,3)$-Theorem and Roth's theorem.

Theorem 3.2. *Induced matching theorem $\implies$ $(6,3)$-Theorem.*

Proof. Given a 3-uniform n-vertex $\mathcal{H}$ that is $(6,3)$-free, we want to show $e(\mathcal{H}) = o(n^2)$. Again by passing to a subgraph with highest average degree, we may assume $\mathcal{H}$ is linear. Take a random equipartition $V(\mathcal{H}) = V_1 \cup V_2 \cup V_3$, then the number of cross edges e_{cr} is in expectation $\frac{2}{9}e(\mathcal{H})$. Choose one partition so that $e_{cr} \geq \frac{2}{9}e(\mathcal{H})$. Define the shadow graph G on $[V_2, V_3]$: $E(G) = \{yz : xyz \in E(\mathcal{H}), x \in V_1, y \in V_2, z \in V_3\}$, namely G is the union of all link graphs of vertices in V_1. For each $v \in V_1$, let M_v be its link graph. Since $\mathcal{H}$ is linear, M_v is a matching and $e(G) = |\cup M_v| = e_{cr} \geq \frac{2}{9}e(\mathcal{H})$. Notice that M_v has to be an induced matching, since otherwise there are two edges u_1v_1 and u_2v_2 in M_v such that $\{u_1, v_1, u_2, v_2\}$ induces a third edge from some $M_{v'}$ with $v' \neq v$. Then we have 3 edges in $\mathcal{H}$ on 6 vertices $\{v, v', v_1, v_2, u_1, u_2\}$. Thus G is a union of $|V_1|$ induced matchings, and by Theorem 3.1, $e(\mathcal{H}) \leq 9e(G)/2 = o(n^2)$. □

Exercise 3.3. Deduce Roth's theorem from induced matching theorem.

Proof of Theorem 3.1. Suppose to the contrary that there is an n-vertex graph G that is a union of n induced matchings and $e(G) > cn^2$. Apply regularity lemma on G with $\varepsilon = c/10$ and $m = 1/\varepsilon$, let $R = R(\varepsilon, 2\varepsilon)$ be a reduced graph corresponding to the regular partition obtained. Do the standard cleaning to get the graph $G_R \subseteq G$ with $e(G_R) \geq e(G) - 3\varepsilon n^2 \geq cn^2/2$ (i.e. deleting edges inside each cluster V_i and edges between sparse or irregular pairs).

By the Pigeonhole Principle, there exists an induced matching M with more than $cn/2$ edges, namely $|V(M)| \geq cn$. Define $U_j = V_j \cap V(M)$ for each $i \in [r] = V(R)$ and set

$$U = \cup\{U_j : |U_j| \geq \varepsilon|V_j|\}.$$

From $\cup U_j$ to U, we have removed at most $\varepsilon n = cn/10$ vertices from $V(M)$, thus $|U| > \frac{9}{10}cn$. Since $U \subseteq V(M)$ and $|U| > |M|$, U spans an edge in M. We know, after the cleaning, this edge has to go between some (U_1, U_2) in some ε-regular pair (V_1, V_2) with density at least 2ε. Since $|U_i| \geq \varepsilon|V_i|$, for $i = 1, 2$, by regularity $d(U_1, U_2) \geq \varepsilon$. This implies $e(U_1, U_2) > \varepsilon|U_1||U_2| > |U_1|$, which means there is a non-M-edge in $[U_1, U_2]$, thus M is not an induced matching, a contradiction. □

We end this section with an old well-known conjecture. The simplest open case is $(7, 4)$.

Conjecture 3.4 (Brown-Erdős-Sós 1973). *Let $s \in \mathbb{N}$. If an n-vertex 3-uniform $\mathcal{H}$ is $(s+3, s)$-free, then*

$$e(\mathcal{H}) = o(n^2).$$

4 Ramsey-Turán problem for K_4

In this section, we present an application of the regularity lemma in Ramsey-Turán problem. Recall that Turán's theorem states that among all n-vertex K_{r+1}-free graphs, the Turán graph $T_r(n)$ has the largest size. Notice that these Turán graphs have rigid structures, in particular, there are independent sets of size linear in n. It is then natural to ask what happens when there is no such big holes. Such problems, first introduced by Sós in 1969, are the substance of the Ramsey-Turán theory.

Given a graph H and natural numbers $m, n \in \mathbb{N}$, the *Ramsey-Turán number* for H is:

$$\mathsf{RT}(n, H, m) := \max\{e(G) : \ |G| = n, \ \alpha(G) \leq m, \text{ and } G \text{ is } H\text{-free}\}.$$

The most classical case is when m is sublinear in n, i.e. $m = o(n)$.

Definition 4.1. Given a graph H, let

$$\varrho(H) := \lim_{\delta \to 0} \lim_{n \to \infty} \frac{\mathsf{RT}(n, H, \delta n)}{\binom{n}{2}}.$$

Define

$$\mathsf{RT}(n, H, o(n)) = \varrho(H) \cdot \binom{n}{2} + o(n^2).$$

A simple averaging argument shows that the limits in the above definition exist.

Exercise 4.2. Prove that $\mathsf{RT}(n, K_3, o(n)) = o(n^2)$.

When there is no restriction on the independence number, recall that $\text{ex}(n, K_4) = n^2/3 \pm O(1)$. In comparison, we have the following.

Theorem 4.3 (Szemerédi 73). $\mathsf{RT}(n, K_4, o(n)) \leq n^2/8 + o(n^2)$.

Sketch of proof. Let G be an n-vertex K_4-free graph with $\alpha(G) = o(n)$. Let R be a weighted reduced graph of G. It suffices to show that R is triangle-free and no edge in R has density larger than $1/2$. Indeed, K_3-free implies that, as a graph, R has at most $r^2/4$ edges; each edge having weight at most $1/2 + o(1)$ implies that, as a weighted graph, $e(R) \le r^2/8 + o(r^2)$, and hence $e(G) \le n^2/8 + o(n^2)$ as desired.

Suppose R has a triangle ijk. Consider the corresponding pairwise dense regular triple V_i, V_j, V_k in G. We can find two typical adjacent vertices $v_iv_j \in E(G)$ with $v_i \in V_i$ and $v_j \in V_j$, having linear codegree in V_k: $|N(v_i) \cap N(v_j) \cap V_k| = \Omega(n)$. As $\alpha(G) = o(n)$, there is an edge in $N(v_i) \cap N(v_j) \cap V_k$, yielding a copy of K_4, a contradiction.

Suppose R has a chubby edge ij, and so $d(V_i, V_j) \ge 1/2 + \Omega(1)$. Then any two typical vertices in V_i has codegree $2(n/2 + \Omega(n)) - n = \Omega(n)$ linear in V_j. This also yields a K_4, as almost all vertices (hence linear many) in V_i are typical, we can find two adjacent ones and pick an edge in their coneighbourhood in V_j, again reaching a contradiction. □

An ingenious geometric construction of Bollobás and Erdős later yields a matching lower bound:

$$\mathsf{RT}(n, K_4, o(n)) = \frac{n^2}{8} + o(n^2).$$

In general, we do not have a Erdős-Simonovits-Stone type theorem for the Ramsey-Turán number $\mathsf{RT}(n, H, o(n))$. The simplest open case is the following.

Open problem 4.4. *Is* $\mathsf{RT}(n, K_{2,2,2}, o(n)) = o(n^2)$*?*