Lecture 4. Turán problem 4, Random algebraic construction

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As mentioned previously, Kövári-Sós-Turán showed that $\mathrm{ex}(n, K_{s,s}) = O(n^{2-1/s})$. A matching lower bound is known only for $K_{s,t}$-free graphs with $t \gg s$ (norm graphs). Recently Bukh used a random algebraic construction to get a new dense $K_{s,t}$-free graphs. We will study this random algebraic construction. In the following subsections, we first explain why the natural construction using $G(n,p)$ fails; then show the dense $K_{s,t}$-free construction of Bukh; and sketch the proof of a breakthrough of Bukh and Conlon on rational Turán exponents.

For a set of vertices U in a graph G, we will write $N^*(U) = \cap_{v\in U} N(v)$ for the common neighbourhood.

1 Long smooth tail in binomial random construction

In this subsection, we briefly go through the construction using $G(n,p)$ and see why it fails to provide a good construction for $\mathrm{ex}(n, K_{s,t})$.

Let us build a bipartite (n,n)-vertex graph G with each pair being an edge with probability p independent of others. To have the desired density, we take $p = n^{-1/s}$, so the expected number of edges is $n^{2-1/s}$ and by standard concentration bound, $e(G) \geq \frac{1}{2}n^{2-1/s}$ with high probability.

To be $K_{s,t}$-free, we need to show $\mathsf{Pr}(K_{s,t} \subseteq G) \to 0$ for some t. To bound this probability, consider an s-set U in a partite set. As every vertex in the other partite set falls in $N^*(U)$ with probability $p^{|U|} = 1/n$, $|N^*(U)|$ is a binomial random variable $B(n, 1/n)$ and distributed roughly as a Poisson random variable with mean 1. In particular,

$$\mathsf{Pr}(|N^*(U) \geq t|) \leq 1/t!. \tag{1}$$

Then by union bound, $\mathsf{Pr}(K_{s,t} \subseteq G) \leq 2\binom{n}{s} \cdot \frac{1}{t!}$, which is close to 0 if $t \geq 10s\frac{\log n}{\log\log n}$. The order of t cannot be improved; it can be shown that with high probability this random graph contains a $K_{s,t}$ with $t = \frac{s}{10}\frac{\log n}{\log\log n}$.

To conclude, though the random variable $|N^*(U)|$ has mean 1, its distribution has a long smooth tail (1). It is therefore likely that $|N^*(U)|$ is large as there are many choice of U.

2 Random algebraic construction

We first build a graph using a random polynomial over $\mathbb{F}_q$ and then get rid of (few) copies of $K_{s,t}$ by deletion method. Roughly speaking, this graph locally enjoys the independence as that of an Erdős-Renyi random graph and so we can estimate its size and also the probabilities of appearance of small subgraph, see Lemmas 2.1 and 2.2. On the other hand, as we use a random polynomial, we can then view $N^*(U)$, the common neighbourhood of an s-set U, as a variety. Note that in the deletion part, we need to get rid of those Us with large $N^*(U)$. What is remarkable now is that, when viewed as a variety, by the Lang-Weil bound from algebraic geometry, the attainable values of $|N^*(U)|$ have a "discontinuity", that is, $|N^*(U)|$ is either bounded, or at least $q/2$. Then, compared to (1), we can get a much better control on the probability of U being a bad s-tuple, as now $\Pr[|N^*(U)| > C] = \Pr[|N^*(U)| > q/2]$. Here $C = O(1)$ and $q \to \infty$.

Let $s \geq 4$, $d = s^2 - s + 2$, q be a sufficiently large prime power, and $n = q^s$. Let $f : \mathbb{F}_q^s \times \mathbb{F}_q^s \to \mathbb{F}_q$ be a uniform random polynomial on $2s$ variables with degree at most d. Now, let G be an (n, n)-vertex bipartite graph with each partite set being a copy of $\mathbb{F}_q^s$, and $uv \in E(G)$ if and only if f vanishes on (u, v), i.e. $f(u, v) = 0$. As f is uniformly chosen, we have the correct density.

Lemma 2.1. *Let f be a uniform $2s$-variate random polynomial over $\mathbb{F}_q$. Then for any $u, v \in \mathbb{F}_q^s$,*

$$\Pr(f(u, v) = 0) = 1/q.$$

Proof. Let $c \in \mathbb{F}_q$ be the constant term in f and set $g = f - c$. As f is uniformly chosen, c is uniformly distributed in $\mathbb{F}_q$. Then, conditioning on the value of $g(u, v)$, $f(u, v) = 0$ if and only if $c = -g(u, v)$, which happens with probability $1/q$. □

The above lemma in particular implies that $\mathsf{E}(e(G)) = n^2/q = n^{2-1/s}$. Similarly, we can obtain the following. We omit its proof.

Lemma 2.2. *Let f be a uniform $2s$-variate random polynomial over $\mathbb{F}_q$. Let $U, V \subseteq \mathbb{F}_q^s$ be sets of size s and r respectively with $s, r \leq \min(\sqrt{q}, d)$. Then*

$$\Pr(f(u, v) = 0 \text{ for all } u \in U,\ v \in V) = q^{-sr}.$$

Say an s-set U in a partite set is *bad*, if $|N^*(U)| \geq t$. We want to show that there are few bad s-sets and a simple deletion suffices to get the desired graph. Let us estimate the moments of $|N^*(U)|$. Note that we can write $|N^*(U)| = \sum_{v \in \mathbb{F}_q^s} \mathbb{1}_{N^*(U)}(v)$. Then for the d-th moment, we have by linearity of expectation that

$$\begin{aligned} \mathsf{E}\big[|N^*(U)|^d\big] &= \mathsf{E}\Big[\Big(\sum_{v \in \mathbb{F}_q^s} \mathbb{1}_{N^*(U)}(v)\Big)^d\Big] = \mathsf{E}\Big[\sum_{v_1, \ldots, v_d \in \mathbb{F}_q^s} \mathbb{1}_{N^*(U)}(v_1) \cdots \mathbb{1}_{N^*(U)}(v_d)\Big] \\ &= \sum_{v_1, \ldots, v_d \in \mathbb{F}_q^s} \mathsf{E}[\mathbb{1}_{N^*(U)}(v_1) \cdots \mathbb{1}_{N^*(U)}(v_d)] \end{aligned}$$

Note that, by Lemma 2.2, the summand above is exactly q^{-sr} if $\{v_1, \ldots, v_d\}$ has $r \leq d$ distinct vertices. Let $M_r = O_d(1)$ be the number of surjective functions from $[d]$ to $[r]$. Then we can bound the d-th moment:

$$\mathsf{E}[|N^*(U)|^d] = \sum_{r \leq d} \binom{q^s}{r} \cdot M_r \cdot q^{-sr} \leq \sum_{r \leq d} M_r = O_d(1).$$

Notice crucially that, by construction, we can view $N^*(U)$ as an algebraic variety:

$$N^*(U) = \{v \in \mathbb{F}_q^s : \ f(u, v) = 0 \text{ for all } \ u \in U\} = \cap_{u \in U}\{f(u, \cdot) = 0\}.$$

Then the Lang-Weil bound says, as a variety, $N^*(U)$ has size either at most C (depending only on s and d), or at least $q/2$. This discontinuity, together with Markov's inequality, implies

$$\Pr[|N^*(U)| \geq C] = \Pr[|N^*(U)| \geq q/2] = \Pr[|N^*(U)|^d \geq (q/2)^d] \leq \frac{\mathsf{E}[|N^*(U)|^d]}{(q/2)^d} = \frac{O_d(1)}{q^d}. \quad (2)$$

And so the expected number of bad s-sets is at most

$$2\binom{n}{s} \cdot \frac{O(1)}{q^d} = O(q^{s-2}).$$

Here we use d is large compared to s, in particular, $d \geq s^2 - s + 2$. Then deleting one vertex from each such bad s-set results in a loss of $O(q^{2s-2}) = o(\mathsf{E}[e(G)])$ negligible number of edges. Taking $t = C + 1$, we have thus obtain a $K_{s,t}$-free graph on at most $2n$ vertices with $\frac{1}{4}n^{2-1/s}$ edges as desired.

3 Rational Turán exponents

A rational number $r \in (1, 2)$ is a *Turán exponent* if there exists a bipartite graph H such that $\mathrm{ex}(n, H) = \Theta(n^r)$. A long-standing open problem in extremal graph theory reads as follows.

Conjecture 3.1. *Every rational number between* 1 *and* 2 *is a Turán exponent.*

Recently, Bukh and Conlon use the random algebraic construction to show that if we are allowed to forbid a finite set of graphs, then every rational in $(1, 2)$ can be realised as an exponent.

Theorem 3.2. *For every rational $r \in (1, 2)$, there exists a finite family $\mathcal{H}$ of graphs such that $\mathrm{ex}(n, \mathcal{H}) = \Theta(n^r)$.*

The forbidden family $\mathcal{H}$ comes from blowups of rooted trees.

Definition 3.3. A *rooted tree* (T, R) consists of a tree T with an independent set R as *roots*. The *density* ρ_T of (T, R) is

$$\frac{e(T)}{|T| - |R|}.$$

For a set of unrooted vertices $S \subseteq V(T) \setminus R$, define its *density* ρ_S to be $\frac{e(S)}{|S|}$, where $e(S)$ is the number of edges in T incident to S. We call the rooted tree (T, R) *balanced* if $\rho_S \geq \rho_T$, for every $S \subseteq V(T) \setminus R$.

Given a rooted tree (T, R), the *p-th power* $\mathcal{T}^p$ of (T, R) is the family of all possible unions of p distinct labelled copies of T, all of which agree on the set of roots R.

By definition, if $|R| \geq 2$, then every leaf in T is a root and leaves are evenly distributed in balanced rooted trees. Note also that the power $\mathcal{T}^p$ contains many different graphs as the unrooted vertices from different copies of T could overlap in every possible ways.

We first show the upper bound, which follows from an averaging argument. Note that the upper bound works for all blowups of rooted trees, not just balanced ones.

Lemma 3.4 (Upper bound)**.** *Let (T, R) be a rooted tree with at least one root, then $\mathrm{ex}(n, \mathcal{T}^p) = O_p(n^{2-1/\rho_T})$.*

Proof. Let $t = |T|$ and G be an n-vertex graph with cn^{2-a} edges, where $a = 1/\rho_T$ and $c > 2(t+p)$. Recall that we can find a subgraph $H \subseteq G$ with $\delta(H) \geq d(G)/2 = cn^{1-a}$. Say $|H| = h \leq n$. By greedily embedding one vertex at a time, we see that the number of labelled copies of unrooted tree T is at least

$$|H| \cdot \delta(H) \cdot (\delta(H) - 1) \cdots (\delta(H) - t + 2) \geq \left(\frac{c}{2}\right)^{t-1} hn^{(t-1)(1-a)}.$$

As there are at most $h^{|R|}$ many choices for the roots R, there is a choice of roots R' in at least

$$\frac{(c/2)^{t-1} hn^{(t-1)(1-a)}}{h^{|R|}} \geq \frac{(c/2)^{t-1} n^{(t-1)(1-a)}}{n^{|R|-1}} = \left(\frac{c}{2}\right)^{t-1} \geq p,$$

where the equality follows from $a = 1/\rho_T = \frac{t-|R|}{t-1}$. These at least p copies of T sitting on R' form a graph in $\mathcal{T}^p$. ☐

To have a matching lower bound, we will have to impose that the rooted tree is balanced.

Exercise 3.5. Give an example of an unbalanced rooted tree (T, R), for which $\mathrm{ex}(n, \mathcal{T}^p) = \Omega_p(n^{2-1/\rho_T})$ is not true.

Lemma 3.6 (Lower bound)**.** *For any balanced rooted tree (T, R), there exists $p \in \mathbb{N}$ such that $\mathrm{ex}(n, \mathcal{T}^p) = \Omega_p(n^{2-1/\rho_T})$.*

Thus, to prove Theorem 3.2, it remains to find a balanced rooted tree (T, R) with $2-1/\rho_T = r$ for each rational $r \in (1, 2)$. We can e.g. take $r = 2 - a/b$ in the following example (which looks like a caterpillar).

Example 3.7. Take $a, b \in \mathbb{N}$ with $a - 1 \le b < 2a - 1$, and let $i = b - a$. Let $T_{a,b}$ be the rooted tree obtained as follows. Take a unrooted path on $[a]$ and then add an additional rooted leaf to each of the $i + 1$ vertices

$$1,\ \lfloor 1 + a/i \rfloor,\ \lfloor 1 + 2 \cdot a/i \rfloor, \ldots,\ \lfloor 1 + (i-1) \cdot a/i \rfloor,\ a.$$

For $b \ge 2a - 1$, define $T_{a,b}$ recursively by attaching a rooted leaf to each unrooted vertex of $T_{a,b-a}$. Note that $T_{a,b}$ has a unrooted vertices and b edges, so $\rho_T = b/a$.

We now sketch a proof for the lower bound Lemma 3.6. It follows closely the construction in Section 2.

Sketch of proof for Lemma 3.6. Let (T, R) be a rooted tree with a unrooted vertices, $r = |R|$ rooted vertices, and b edges. Let $s = 2br, d = sb$ and $n = q^b$ for large prime power q. Take $2b$-variate independent uniform random polynomials $f_1, \ldots, f_a : \mathbb{F}_q^b \times \mathbb{F}_q^b \to \mathbb{F}_q$, each of degree at most d. Define (n, n)-vertex bipartite G with each partite set being $\mathbb{F}_q^b$, where $uv \in E(G)$ if and only if all of f_i vanish on (u, v), i.e.

$$f_1(u, v) = \ldots = f_a(u, v) = 0.$$

As $f_1, \ldots, f_a$ are independent, similar to Lemma 2.1, we see that the edge density is q^{-a} and so the expected number of edges in G is $n^{2-a/b}$. We are left to show that the expected number of copies of graphs in $\mathcal{T}^p$ is negligible.

Fix now vertices $w_1, \ldots, w_r$ in G and let $\mathcal{U}$ be the collection of copies of T in G rooted at $\{w_1, \ldots, w_r\}$. We need to bound the moments of $|\mathcal{U}|$ (instead of moments of $|N^*(U)|$ in Section 2). One can similarly show that $\mathsf{E}(|\mathcal{U}|^s) = O_s(1)$ and use the Lang-Weil bound to show that either $|\mathcal{U}| \le C$ (depending only on T) or $|\mathcal{U}| \ge q/2$. Thus, as before,

$$\mathsf{Pr}(|\mathcal{U}| > C) = \mathsf{Pr}(|\mathcal{U}| \ge q/2) = \frac{O_s(1)}{(q/2)^s}.$$

Consequently, the number of *bad* $\{w_1, \ldots, w_r\}$ (i.e. sitting in more than C copies of T as roots) is at most

$$2n^r \cdot \frac{O_s(1)}{(q/2)^s} = o(1),$$

and we can remove one vertex from each of these bad choices to get the desired dense $\mathcal{T}^p$-free graph with $p = C + 1$. □

The rational Turán exponent conjecture, Conjecture 3.1, remains open. Given a rooted tree (T, R) and $p \in \mathbb{N}$. let T^p be its non-degenerate p-blowup, that is, T^p is obtained by taking p copies of T and letting them agree on the roots but disjoint otherwise. Thanks to Lemma 3.6, the following conjecture of Bukh and Conlon would imply Conjecture 3.1.

Conjecture 3.8. *For any balanced rooted tree* (T, R), $\mathrm{ex}(n, T^p) = O_p(n^{2-1/\rho_T})$.

Another related problem was raised recently by Kang, Kim and Liu. They showed that the following conjecture about 1-subdivision of bipartite graph would imply Conjecture 3.1. For a graph F, let $\mathrm{sub}(F)$ be the 1-*subdivision* of F, obtained from F by replacing all edges of F with pairwise internally disjoint paths of length two.

Conjecture 3.9 (Subdivision conjecture)**.** *Let F be a bipartite graph. If* $\mathrm{ex}(n, F) = O(n^{1+\alpha})$ *for some* $\alpha > 0$, *then*

$$\mathrm{ex}(n, \mathrm{sub}(F)) = O(n^{1+\frac{\alpha}{2}}).$$