Lecture 3. Turán problem 3

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1 Even cycles and BFS argument

Besides the complete bipartite graph $K_{s,t}$, another class of bipartite graphs that attracts lots of attention is even cycles C_{2k}. For the upper bound, Bondy and Simonovits proved that $\mathrm{ex}(n, C_{2k}) = O(n^{1+1/k})$. We shall give a proof of this shortly. The matching lower bound $\Omega(n^{1+1/k})$ is only known for $k = 2, 3, 5$ coming from constructions using finite geometry. The first open case is 8-cycle, where the lower bound is due to Lazebnik, Ustimenko and Woldar.

Open problem 1.1. *Improve* $\Omega(n^{6/5}) \le \mathrm{ex}(n, C_8) \le O(n^{5/4})$.

Another interesting question is to construct dense asymmetric graphs without short even cycles. Write $\mathrm{ex}(m, n, H)$ for the maximum number of edges in an H-free subgraph of $K_{m,n}$. It is known that

$$\mathrm{ex}(m, n, C_{2k}) = \begin{cases} O(m^{\frac{1}{k}}(mn)^{\frac{1}{2}} + m + n), & \text{even } k; \\ O((mn)^{\frac{1}{2}(1+\frac{1}{k})} + m + n), & \text{odd } k. \end{cases}$$

Erdős initially conjectured that $\mathrm{ex}(n^{2/3}, n, C_6) = O(n)$, which was disproved by a construction showing $\mathrm{ex}(n^{2/3}, n, C_6) \ge \Omega(n^{16/15})$.

Open problem 1.2. *Is* $\mathrm{ex}(n^{2/3}, n, C_6) = \Omega(n^{10/9})$*?*

Coming back to the promised upper bound, we give a proof of the bound below, following an argument of Pikhurko.

Theorem 1.3. *For large* n, $\mathrm{ex}(n, C_{2k}) \le 4kn^{1+1/k}$.

Before giving his proof, let us mention some recent development. Determining the extremal number for even cycles is another main open problem in this area. People have split opinions: some believe in a matching lower bound $\Omega(n^{1+1/k})$; while some believe that $\mathrm{ex}(n, C_{2k}) = o(n^{1+1/k})$. The best upper bound, due to He, is $O(\sqrt{k \log k}\, n^{1+1/k})$. As a step forward, it would be interesting to show the following bound.

Open problem 1.4. *Is* $\mathrm{ex}(n, C_{2k}) = o(\sqrt{k}\, n^{1+1/k})$*?*

We now present some definitions and lemmas needed for the proof of Theorem 1.3. The first two are basic tools which finds a subgraph which is either bipartite or having high minimum degree. Recall that $d(G)$ is the average degree of a graph G.

Exercise 1.5. Every graph G contains a bipartite subgraph H with $e(H) \ge e(G)/2$.

Exercise 1.6. Every graph G contains a subgraph H with $\delta(H) \ge d(G)/2$.

Definition 1.7. A *Θ_k-graph* is a cycle of length at least $2k$ with a chord.

Given high minimum degree, it is easy to embed such a graph greedily.

Exercise 1.8. Let $k \geq 3$ and H be a bipartite graph with $d(H) \geq 2k$. Then H contains a Θ_k-graph.

The following lemma shows that Θ_k-graphs contain paths of varying lengths.

Lemma 1.9. *Let F be a Θ_k-graph and $A \cup B$ be a non-trivial partition of $V(F)$. If F is not bipartite with bipartition $A \cup B$. Then there are A, B-paths of all lengths less than $|F|$.*

Sketch of proof. We shall prove the contrapositive. That is, fixing $\ell < n = |F|$, if there is no A, B-path of length ℓ, then F is bipartite with bipartition $A \cup B$.

Identify $V(F)$ with $\mathbb{Z}_n$ so that i is adjacent to $i \pm 1$ for all $i \in \mathbb{Z}$. Think of the partition $A \cup B$ as a 2-colouring c of $\mathbb{Z}_n$. Define the set of periods of c:

$$P = \{m \in \mathbb{Z}_n : \ \forall i \in \mathbb{Z}_n, \ c(i) = c(i+m)\}.$$

By assumption, $\ell \in P$. Note that as $A \cup B$ is a non-trivial partition, the smallest period $m \in P$ divides n and furthermore $P = \{mi : i \in \mathbb{Z}_n\}$. We may assume $m > 2$ for otherwise $A \cup B$ is a bipartition of F as desired.

Now, say the chord in F connects vertices 0 and r. If $n - r \equiv r \equiv 1 \mod m$, then $n \equiv 2 \mod m$, contradicting $m > 2$ and $m \mid n$. Say, then, $r - 1 \notin P$. Recall that $\ell \in P$, so there must exist some $j \in \mathbb{Z}_n$ such that $c(j) \neq c(j + \ell + r - 1)$. We may further assume $-m < j \leq 0$. Then the ℓ-walk

$$j, j+1, \ldots, -1, 0, r, r+1, \ldots, j + \ell + r - 1$$

is from A to B. This is a contradiction unless $\ell + r - 1 \geq n$. The rest cases can be handled similarly. □

The idea of the proof for Theorem 1.3 is to take a Breadth First Search (BFS) tree T. If there is a Θ_k-subgraph between any of the first k pairs of two consecutive layers of T, then we can find a C_{2k} using Lemma 1.9. Then, by Exercise 1.8, lacking Θ_k-subgraph implies that each of the first k layers expands by a factor of $\Omega(d(G)/k)$, yielding the desired bound.

Proof of Theorem 1.3. Let G be an n-vertex C_{2k}-free graph with $e(G) \geq kn^{1+1/k}$. By Exercises 1.5 and 1.6, we can pass to a bipartite subgraph H with $\delta(H) \geq d(G)/4 \geq 2kn^{1/k}$.

Take a vertex x in H and, for $i \geq 0$, let $V_i = N_H^i(x)$ be the set of vertices of distance i from x and denote by $H_i = H[V_i, V_{i+1}]$ the induced bipartite subgraph.

Claim 1.10. *For all $i \in [k-1]$, H_i has no Θ_k-subgraph.*

Proof of claim. Suppose H_i contains a Θ_k-subgraph F, which must be bipartite. Let $Y \cup Z$ be a bipartition of F, and let $T \subseteq H$ be a BFS tree rooted at x. For a vertex $v \in T$, we write $D_T(v)$ its descendents in T.

Let y be a minimal ancestor of $Y_i = Y \cap V_i$, that is, y is the farthest vertex from x such that $Y_i \subseteq D_T(y)$. Let z be a child of y such that $A = D_T(z) \cap Y_i \neq \varnothing$. By the minimality of y, we see that $Y_i \setminus A \neq \varnothing$. Hence, letting $B = (Y \cup Z) \setminus A$, $A \cup B$, clearly non-trivial, is not a bipartition of F.

Now, let ℓ be the distance between x and y. So $\ell < i$ and $2k - 2(i - \ell) < 2k \leq |F|$. Then by Lemma 1.9, there is an a, b-path P of length $2k - 2(i - \ell)$ with $a \in A \subseteq Y_i$ and $b \in B$. As P is of even length, we must have b also in Y_i. Let P_a and P_b be the y, a-path and y, b-path respectively in T, each of length $i - \ell$. By the choices of y and $F \subseteq H_i$, P, P_a and P_b are internally vertex disjoint. Therefore, P, P_a, P_b form a copy of C_{2k}, a contradiction. ■

Thus, by Exercise 1.8, $d(H_i) \leq 2k - 1$. Consequently, as $n \gg k$, for each $i \in [k-1]$, on average, each vertex in V_i sends forward to V_{i+1} at least $\delta(H) - O(k)$ edges, while each vertex in V_{i+1} sends back to V_i at most $2k - 1$ edges. Therefore, for each $0 \leq i \leq k-1$, the ratio $|V_{i+1}|/|V_i| \geq \frac{\delta(H) - O(k)}{2k-1}$, implying that $(\frac{\delta(H) - O(k)}{2k-1})^k \leq n$, or $\delta(H) < 2kn^{1/k}$, a contradiction. □

2 Supersaturation

By the definition of extremal number, if an n-vertex graph has more than $\mathrm{ex}(n, H)$ number of edges, then there must be a copy of H in G. The *supersaturation* phenomenon refers to the situation that when the size of a graph goes beyond the extremal bound, then often time, not just one, but many copies of forbidden structures are guaranteed to appear. This is called Erdős-Rademacher problem when H is non-bipartite. Here is an example.

Proposition 2.1 (Rademacher)**.** *Every n-vertex graph with $\frac{n^2}{4}+1$ edges contains at least $\lfloor \frac{n}{2} \rfloor$ triangles.*

We would like to take a look at the case when H is bipartite, in particular C_4. This is also related to the Sidorenko's conjecture. We will come back to this conjecture when we introduce informatic theoretic methods.

Here, let us see how we can extend the double counting and convexity argument in bounding $\mathrm{ex}(n, C_4)$ to get the following supersaturation result for C_4 for graphs with at least $2\,\mathrm{ex}(n, C_4)$ edges.

Theorem 2.2. *Let G be an n-vertex graph with n large and $d(G) \geq 2\sqrt{n}$. Then there are at least $\frac{d(G)^4}{8}$ copies of C_4 in G.*

Proof. Let $d = d(G)$. For vertices u, v, write $d_{u,v} = |N(u) \cap N(v)|$ for their codegree. As before, by Jensen's inequality, the number of cherries $K_{1,2}$ in G is at least

$$\sum_{v \in V(G)} \binom{d(v)}{2} \geq n \binom{\frac{1}{n}\sum d(v)}{2} = n\binom{d}{2}.$$

Then we see that the average codegree in G is

$$d_{co} = \frac{1}{\binom{n}{2}} \sum_{u,v \in \binom{V(G)}{2}} d_{u,v} = \frac{\# K_{1,2}}{\binom{n}{2}} \geq \frac{d(d-1)}{n-1} \geq 2.$$

As every C_4 can be obtained by picking two common neighbours of a pair of vertices, applying again Jensen's inequality, we have the number of C_4 in G is at least

$$\sum_{u,v \in \binom{V(G)}{2}} \binom{d_{u,v}}{2} \geq \binom{n}{2}\binom{d_{co}}{2} \geq \frac{d^4}{8}.$$

□

3 Regularisation

In this section, we present a useful lemma of Erdős and Simonovits, which allows us to assume the host graph is almost regular when working on bipartite Turán problem. We say a graph G is K*-almost regular* if $\Delta(G) \leq K \cdot \delta(G)$.

Lemma 3.1. *Let $0 < \varepsilon < 1$ and $c > 0$ and n be sufficiently large. Let G be an n-vertex graph $e(G) \geq cn^{1+\varepsilon}$. Then G contains an m-vertex K-almost regular subgraph H with $m \geq n^{\frac{\varepsilon-\varepsilon^2}{4+4\varepsilon}}$ and $e(H) \geq \frac{2c}{5} m^{1+\varepsilon}$, where $K = 20 \cdot 2^{\frac{1}{\varepsilon^2}+1}$.*

Its proof is not long, but we do not see the need to include all details.

Sketch of proof. Let $p = 2^{\frac{1}{\varepsilon^2+1}} = \frac{K}{20}$ and take a $2p$-equipartition $V_1 \cup \ldots \cup V_{2p}$ of $V(G)$ with V_1 containing the highest degree vertices.

If at most half of the edges are incident to V_1, we say G is of type 1. In this case, delete V_1 and then repeatedly delete vertices of degree less than $\frac{c}{10}n^{\varepsilon}$ until no such vertex exists. The resulting graph H is as desired.

If more than half of the edges are incident to V_1, we call G type 2. Then by pigeonhole, we can find V_i such that $G_1 = G[V_1, V_i]$ has at least $\frac{1}{2p}$ proportion of edges incident to V_1. Now we iterate the analysis. If G_1 is of type 1, we are done; if not, then find a subgraph in G_1 as above and repeat. I can be shown that this process terminates at a large type 1 subgraph. □

4 Cube and graphs with bounded degeneracy

Yet, another major open problem is about graphs with bounded degeneracy. A graph is k-*degenerate* if every subgraph of it contains a vertex of degree at most k. The best known upper bound for k-degenerate graphs is $\mathrm{ex}(n, H) = O(n^{2-\frac{1}{4k}})$ due to Alon-Krivelevich-Sudakov.

Conjecture 4.1. *Let H be a k-degenerate bipartite graph. Then* $\mathrm{ex}(n, H) = O(n^{2-1/k})$.

The special case when one side of H has maximum degree k can be proved by dependent random choice. Conlon and Lee conjectured that such bound is only tight when H contains $K_{k,k}$.

Conjecture 4.2. *Let H be a $K_{k,k}$-free bipartite graph with maximum degree k on one side, then* $\mathrm{ex}(n, H) = O(n^{2-1/k-\Omega(1)})$.

They solved the $k = 2$ case via studying 1-subdivision of complete graphs. For general k, the best bound is $o(n^{2-1/k})$ due to Sudakov-Tomon. We sugguest hypercubes as a test case. Write Q_k for the k-dimensional hypercube. Note that Q_k is k-regular and $K_{3,3}$-free.

Open problem 4.3. *Is it true that* $\mathrm{ex}(n, Q_k) \le O(n^{2-1/k-\Omega(1)})$.

In particular, the above is true for 3-dimensional cube. We shall in fact prove a bound for Q_3^+, the graph obtained from adding the long diagonal to cube. Note that to embed Q_3^+, it suffices to find an edge xy and then embed a C_6 between their neighbourhoods $G_{x,y} = G[N(x), N(y)]$. The idea is to use supersaturation for C_4 to find a 'heavy' edge which sits in many copies of C_4. This implies that $G_{x,y}$ has many edges and must then contain a C_6.

Theorem 4.4. $\mathrm{ex}(n, Q_3^+) = O(n^{8/5})$.

Proof. By Lemma 3.1, we may assume that our graph is almost regular. So take $0 < 1/C \ll 1/K \ll 1$ and let G be an n-vertex K-almost regular graph with average degree $d = Cn^{3/5}$. By Theorem 2.2, there are at least $d^4/8$ many C_4. Thus, there is an edge xy sitting in

$$\frac{d^4/8}{e(G)} = \frac{d^4/8}{dn/2} = \frac{d^3}{4n} = \frac{C^3}{4}n^{4/5} > \mathrm{ex}(2Kd, C_6)$$

many C_4. Note that the number of C_4 containing xy is precisely the number of edges in $G_{x,y} = G[N(x), N(y)]$. As $|G_{x,y}| \le 2Kd$, there is a C_6 in $G_{x,y}$, which together with x, y, forms a copy of Q_3^+. □

Sadly, we do not know the order of magnitude for $\mathrm{ex}(n, Q_3)$. The best known lower bound comes from dense C_4-free graphs.

Open problem 4.5. *Improve* $\Omega(n^{3/2}) \le \mathrm{ex}(n, Q_3) \le O(n^{8/5})$.