Lecture 16. The number of maximal triangle-free graphs

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In this lecture, we will introduce informally the hypergraph container theorem and give an application of it on counting maximal triangle-free graphs.

1 Hypergraph container

The hypergraph container method, developed recently by Balogh-Morris-Samotij, and independently Saxton-Thomason, extends the graph container theorem of Kleitman and Winston to hypergraph settings. It had quickly become an important tool in extremal and probabilistic combinatorics. The method has been also applied in other areas such as discrete geometry, number theory etc. The formal statement is technical; for the first encounter, the following informal one is more illuminating. As in the graph case, it says that we can group independent sets in certain hypergraphs into 'few' containers, each of which is nearly independent.

Theorem 1.1 (Hypergraph container theorem (Informal))**.** *Let $\mathcal{H}$ be a uniform hypergraph. If its edges are evenly distributed, then there are a family of containers $\mathcal{C} \subseteq 2^{V(G)}$ such that*

- *for every independent set I in $\mathcal{H}$, there exists $C \in \mathcal{C}$ s.t. $I \subseteq C$;*
- *the number of containers $C \in \mathcal{C}$ is small;*
- *each container $C \in \mathcal{C}$ is almost independent, i.e. $e(\mathcal{H}(C))$ is small.*

At a first glance, it is not easy to see how we can use such a statement. In fact, many problems can be rephrased as one about independent sets in hypergraphs.

1.1 Containers for triangle-free graphs

Consider the 3-uniform hypergraph $\mathcal{H} = (V, E)$ where

- vertices in $\mathcal{H}$ are edges in K_n: $V = E(K_n)$; and
- each triangle in K_n gives rise to an edge in $\mathcal{H}$.

Then independent sets in $\mathcal{H}$ are precisely the family of triangle-free graphs on $[n]$. If we tailor Theorem 1.1 for this hypergraph, we obtain the following.

Theorem 1.2 (Containers for triangle-free graphs)**.** *There are a family of graphs $\mathcal{F} \subseteq 2^{\binom{[n]}{2}}$ satisfying the following:*

- *for every triangle-free graph G on $[n]$, there exists $F \in \mathcal{F}$ containing G as subgraph: $G \subseteq F$;*
- *there are at most $2^{n^{3/2+o(1)}}$ graphs in $\mathcal{F}$;*
- *each graph $F \in \mathcal{F}$ is almost triangle-free, it has at most $o(n^3)$ triangles.*

Recall that the triangle removal lemma states that if an n-vertex graph has $o(n^3)$ triangles, then it can be made triangle-free by removing $o(n^2)$ edges. As each container is almost triangle-free, by triangle removal lemma and Mantel's theorem, we see that

- each container $F \in \mathcal{F}$ for triangle-free graphs has at most $(\frac{1}{4} + o(1))n^2$ edges.

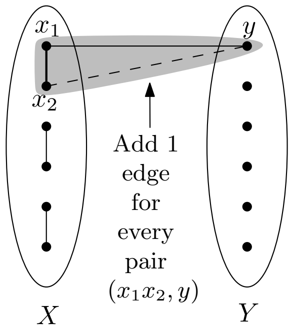

Figure 1: Lower bound contruction for maximal triangle-free graphs.

2 Counting maximal triangle-free graphs

How many triangle-free graphs on vertex set $[n]$ are there[1]? Here is an easy way to generate many triangle-free graphs: take a balanced complete bipartite graph $K_{n/2,n/2}$; as all of its subgraphs are triangle-free, we get $2^{n^2/4}$ triangle-free graphs. A classical result of Erdős, Kleitman and Rothschild in the 70s shows that this bound is optimal up to an exponential error: there are $2^{(1/4+o(1))n^2}$ triangle-free graphs on $[n]$. They further determined the typical structure, that is, almost all of the triangle-free graphs are bipartite.

As a warm up, let us see how to use container theorem to get the Erdős-Kleitman-Rothschild bound. Take the family of containers $\mathcal{F}$ for triangle-free graphs in Theorem 1.2. As every triangle-free graph is a subgraph of some container, we can bound the number of triangle-free graphs from above by the total number of subgraphs in all containers:

$$\sum_{F\in\mathcal{F}} 2^{e(F)} \leq |\mathcal{F}| \cdot 2^{(1/4+o(1))n^2} \leq 2^{cn^{3/2}\log n} \cdot 2^{(1/4+o(1))n^2} = 2^{(1/4+o(1))n^2}.$$

A triangle-free graph G is *maximal* triangle-free if it is no longer triangle-free after adding any non-edge. Notice that the triangle-free graphs we obtain above are subgraphs of $K_{n/2,n/2}$, and thus not maximal triangle-free. It is then natural to wonder if there are a lot fewer maximal triangle-free graphs.

Problem 2.1 (Erdős)**.** Estimate the number of maximal triangle-free graphs.

Let us first describe a lower bound construction yielding $2^{n^2/8}$ maximal triangle-free graphs on $[n]$. Assume that n is a multiple of 4.

- Start with a graph on a vertex set $X \cup Y$ with $|X| = |Y| = n/2$ such that X induces a perfect matching and Y is an independent set (see Figure 1).

- For each pair of a matching edge x_1x_2 in X and a vertex $y \in Y$, add exactly one of the edges x_1y or x_2y. Since there are $n/4$ matching edges in X and $n/2$ vertices in Y, we obtain $2^{n^2/8}$ triangle-free graphs.

- These graphs may not be maximal triangle-free, but since no further edges can be added between X and Y, all of these $2^{n^2/8}$ graphs extend to distinct maximal ones.

Balogh and Petříčková recently proved a matching upper bound.

Theorem 2.2. *The number of maximal triangle-free graphs on vertex set* $[n]$ *is*

$$2^{n^2/8+o(n^2)}.$$

[1]We are counting labeled graphs here.

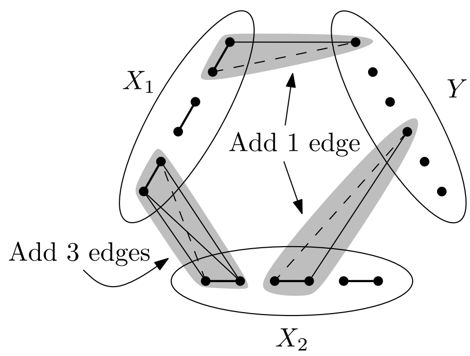

Figure 2: Lower bound contruction for maximal K_4-free graphs.

This result was then extended by Balogh-Liu-Petříčková-Sharifzadeh as follows, showing that a typical maximal triangle-free graph looks like the one in the construction above.

Theorem 2.3. *For almost every maximal triangle-free graph G on $[n]$, there is a vertex partition $X \cup Y$ such that $G[X]$ is a perfect matching and Y is an independent set.*

We end this section with an open problem.

Conjecture 2.4. *For each $r \geq 2$, the number of maximal K_{r+1}-free graphs on vertex set $[n]$ is*

$$2^{\frac{1}{2}\text{ex}(n,K_{r+1})+o(n^2)}.$$

Even the case of maximal K_4-free is open. Here is a construction giving the lower bound in the above conjecture. Assume that n is a multiple of $2r$.

- Partition the vertex set $[n]$ into r equal classes $X_1, \dots, X_{r-1}, Y$, and place a perfect matching into each of $X_1, \dots, X_{r-1}$ and leave Y empty (see Figure 2 for $r = 3$ case).
- Between the vertices of two matching edges from different classes X_i and X_j place exactly three edges, and between a vertex in Y and a matching edge in X_i put exactly one edge. Then extend each graph to a maximal one.

3 Proof of Theorem 2.2

We will present a proof of Theorem 2.2 using Theorem 1.2. We need also the following result of Hujter and Tuza. Note that the bound below is optimal by considering the graph consists of disjoint edges.

Theorem 3.1. *Every an n-vertex triangle-free graph has at most $2^{\frac{n}{2}}$ maximal independent sets.*

Proof of Theorem 2.2. Let $\mathcal{F}$ be the family of containers for triangle-free graphs as in Theorem 1.2. As there are only $2^{o(n^2)}$ containers, it suffices to prove that any $F \in \mathcal{F}$ contains at most $2^{n^2/8+o(n^2)}$ maximal triangle-free graphs.

Fix an arbitrary $F \in \mathcal{F}$. As F is almost triangle-free, by triangle removal lemma, we can partition $F = A \cup B$ so that A is triangle-free (thus $e(A) \leq n^2/4$) and $e(B) = o(n^2)$. We can build a maximal triangle-free graph in the following two steps:

1. choose its (triangle-free) intersection say B' with B;
2. extend B' in A to a maximal triangle-free graph.

Note that the number of choices for Step 1 is at most $2^{e(B)} = 2^{o(n^2)}$. It thus suffices to show that for a fixed $B' \subseteq B$, the number of ways in Step 2 to add edges in A to B' to get a maximal triangle-free graphs is at most $2^{n^2/8+o(n^2)}$.

For bound the number of such extensions, build an auxiliary graph $\mathit{\Gamma}$ with vertex set $E(A)$ in which two vertices are adjacent if they form a triangle with some edges in B'. A crucial observation is that the number of extensions in Step 2 is at most the number of maximal independent sets in $\mathit{\Gamma}$.

We claim that $\mathit{\Gamma}$ is triangle-free. This would finish the proof, as then the number of extensions is, by Theorem 3.1, at most

$$2^{|\mathit{\Gamma}|/2} = 2^{e(A)/2} \leq 2^{n^2/8}.$$

Suppose then there is a triangle xyz in $\mathit{\Gamma}$. Note that x, y, z being pairwise adjacent in $\mathit{\Gamma}$ means that they, as edges in A, have to pairwise share an endpoint. This is only possible if x, y, z form a triangle-free or a star in A. In the former case, x, y, z should have been an independent set in $\mathit{\Gamma}$. In the latter case, say the star has center u and leaves v_1, v_2, v_3. Then by the definition of $\mathit{\Gamma}$, v_1, v_2, v_3 form a triangle in B', a contradiction. $\square$