Lecture 11. Extremal set theory, part 3

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18th May 2021

Today we will prove the upper bound theorem, which implies the fractional Helly theorem. Then we move on to Kruskal-Katona theorem.

1 Fractionally Helly theorem

Let us recall the fractional Helly theorem.

Theorem 1.1 (Fractionally Helly theorem)**.** *Let $\alpha \in (0,1)$ and $\mathcal{F}$ be a collection of convex sets in $\mathbb{R}^d$. If at least $\alpha\binom{|\mathcal{F}|}{d+1}$ many $(d+1)$-tuples $\mathcal{A} \in \binom{\mathcal{F}}{d+1}$ satisfy $\cap\mathcal{A} \neq \varnothing$, then there exists a subcollection $\mathcal{F}' \subseteq \mathcal{F}$ such that $\cap\mathcal{F}' \neq \varnothing$ and $|\mathcal{F}'| \geq \beta|\mathcal{F}|$, where $\beta \geq (1-(1-\alpha)^{\frac{1}{d+1}})$.*

We shall present a proof of the upper bound theorem, Theorem 1.2, which implies the fractional Helly theorem when applied to the nerve complex of a family of convex sets. The *nerve* complex of a family of sets consists of faces corresponding to subfamilies with non-empty common intersection. Taking $r = \beta n - d$ in Theorem 1.2, we get the contrapositive of Theorem 1.1.

To state the upper bound theorem, we need some definitions. A *simplicial complex* K is a set of simplices that satisfies the following conditions:

- Every face of a simplex from K is also in K.
- The non-empty intersection of any two simplices σ_1, σ_2 in K is a face of both σ_1 and σ_2.

We will write $f_j(K)$ for the number of j-dimensional faces in K.

A face σ in a simplicial complex K is *free* if σ is contained in a unique maximal face τ. Given a free face σ with $\tau \supseteq \sigma$ the unique maximal face containing it, denote by $[\sigma, \tau]$ the set of all faces containing σ. An *elementary* (a,b)*-collapse* is the process of removing $[\sigma,\tau]$ for some free face σ with $|\sigma| = a$ and $|\tau| = b$. We say a simplicial complex is *d-collapsible* if all face of size at least d can be removed after some sequence of elementary collapses, in which each elementary collapse is of type (d, d') for some $d' \geq d$, i.e. there exists a collapse process

$$K \to K_1 = K \setminus [\sigma_1, \tau_1] \to K_2 = K_1 \setminus [\sigma_2, \tau_2] \to \ldots \to K_t,$$

such that $f_{d-1}(K_t) = 0$.

Theorem 1.2 (Alon-Kalai 85)**.** *Let K be a d-collapsible simplicial complex on n vertices. If $\dim K < d + r$, i.e. $f_{d+r}(K) = 0$, then for each $d \leq j \leq d+r-1$,*

$$f_j(K) \leq \sum_{i=j+1-d}^{r} \binom{n-i}{d}\binom{i-1}{j-d}.$$

In particular,

$$f_d(K) \leq \binom{n}{d+1} - \binom{n-r}{d+1}.$$

Proof. Given a process of elementary collapses eliminating all faces of size at least d, for each $i \geq 0$, write h_i for the number of elementary $(d, d+i)$-collapses in the process. Note that in each such collapse, precisely $\binom{i}{j+1-d}$ j-dimensional faces were deleted for $d-1 \leq j \leq d+i-1$.

Recall that $f_{d+r}(K) = 0$. For each $0 \leq i \leq r$, write $H_i = \sum_{i'=i}^{r} h_{i'}$. Then for $d \leq j \leq d+r-1$, we have

$$f_j(K) = \sum_{i=j+1-d}^{r} h_i \cdot \binom{i}{j+1-d} = \sum_{i=j+1-d}^{r} (H_i - H_{i+1}) \binom{i}{j+1-d} = \sum_{i=j+1-d}^{r} H_i \cdot \binom{i-1}{j-d}.$$

We are left to show that for each $i \geq 0$,

$$H_i \leq \binom{n-i}{d}.$$

Recall that H_i is the number of elementary (d, d')-collapses, $d' \geq d$, in the process. Let $[\sigma_k, \tau_k]$, $k \in [h]$, be the removed faces corresponding to those in H_i; so $|\tau_k| \geq d+i$. Note that for each $k \in [h]$, we have $\sigma_k \subseteq \tau_k$ and every $k < k'$, $\sigma_k \not\subseteq \tau_{k'}$. Setting $A_k = \sigma_k$ and B_k to be the complement of τ_k (so $|A_k| = d$ and $|B_k| \leq n-d-i$), we can then apply the skewed set-pairs inequality to $\{A_k\}_{k \in [h]}$ and $\{B_k\}_{k \in [h]}$ to get that

$$H_i = h \leq \binom{n-d-i+d}{d} \leq \binom{n-d-i+d}{d} = \binom{n-i}{d},$$

as desired. $\square$

2 Kruskal-Katona theorem

2.1 A motivating problem: triangle maximisation

Consider the following natural problem. Given an n-vertex graph with m edges, how many triangles it must contain? In the limit language, given the edge density, what is the minimum triangle density?[1] This is the famous Erdős-Rademacher clique minimisation problem from the 1940s. This problem is notoriously difficult and its asymptotic solution was discovered only fairly recently by Razborov (triangle case), Nikiforov (K_3, K_4) and Reiher (all cliques). The exact triangle minimisation is solved very recently by Liu, Pikhurko and Staden.

The counterpart of triangle maximisation is equally natural, that is, if an n-vertex graph G has m edges, what is the most number of triangles it can have? This problem turns out to be much easier. The complete answer would follow straightforwardly from the Kruskal-Katona theorem that we would see in a minute. Basically, given the number of edges, the best we can do (to maximise the number of triangles) is to pack edges into a clique.

2.2 Minimising shadow

Definition 2.1. The *shadow* of a k-uniform hypergraph $\mathcal{F} \subseteq \binom{[n]}{k}$, denoted by $\partial\mathcal{F} \subseteq \binom{[n]}{k-1}$, is the $(k-1)$-uniform hypergraph consisting of $(k-1)$-sets that lie in edges of $\mathcal{F}$. That is,

$$\partial\mathcal{F} = \left\{ F \in \binom{[n]}{k-1} : \ F \subseteq G \text{ for some } G \in \mathcal{F} \right\}.$$

Kruskal-Katona theorem studies size of the shadow of a hypergraph.

Problem 2.2. Let $n \geq k$ and $0 \leq m \leq \binom{n}{k}$, what is the minimum shadow size of an n-vertex k-uniform hypergraph with m edges?

[1] Recall that the triangle density in a graph G is $k_3(G)/\binom{n}{3}$, where $k_3(G)$ is the number of triangles in G.

Let us try to construct a k-uniform hypergraph $\mathcal{H}$ with small shadow. Consider the baby case $k = 3$, so the shadow $\partial\mathcal{H}$ is a graph. Since the number of edges $e(\mathcal{H}) = m$ is given, it seems economical to pack edges into cliques. Let $n_3 \in \mathbb{N}$ be such that $\binom{n_3}{3} \le m < \binom{n_3+1}{3}$. Add a clique on a set V_3 of n_3 vertices to $\mathcal{H}$. We have to add $m - \binom{n_3}{3}$ more edges. As there is no room left in V_3, we need to introduce a new vertex v and try to add triples containing v and pairs from V_3, and we want these pairs densely packed inside V_3. So take $V_2 \subseteq V_3$ of size n_2, where $\binom{n_2}{2} \le m - \binom{n_3}{3} < \binom{n_2+1}{2}$, and add all triples $\{v\} \cup e$, $e \in \binom{V_2}{2}$, to $\mathcal{H}$. We are left with adding the remaining $m - \binom{n_3}{3} - \binom{n_2}{2} < \binom{n_2+1}{2} - \binom{n_2}{2} = n_2$ edges to $\mathcal{H}$. For these edges, we can take a vertex $u \in V_3 \setminus V_2$ and add triples containing $\{u, v\}$ and a set $V_1 \subseteq V_2$ of vertices of size $n_1 = m - \binom{n_3}{3} - \binom{n_2}{2}$. In this construction, we see that $|\partial\mathcal{H}| = \binom{n_3}{2} + \binom{n_2}{1} + 1$, where $m = \binom{n_3}{3} + \binom{n_2}{2} + n_1$.

This construction of k-sets in the initial segment of $\mathbb{N}$ is called the *colexicographic order* (colex) of finite subsets of $\mathbb{N}$. The Kruskal-Katona theorem states that the above construction is best possible, i.e. colex ordering minimises the shadow size.

Theorem 2.3 (Kruskal-Katona 1963). *Let $\mathcal{F}$ be a k-uniform hypergraph with $m = \binom{n_k}{k} + \binom{n_{k-1}}{k-1} + \cdots + \binom{n_s}{s}$, then*

$$|\partial\mathcal{F}| \ge \binom{n_k}{k-1} + \binom{n_{k-1}}{k-2} + \cdots + \binom{n_s}{s-1}.$$

We leave as an exercise to show that for any natural number m, we can write it (uniquely!) as a sum of binomial coefficients as above.

2.3 Shifting

We will present a sketch of a proof of Kruskal-Katona using the *shifting/compression* argument, which is an operation that transform our hypergraph towards a colex one without increasing the shadow.

Definition 2.4. Given $\mathcal{F} \subseteq \binom{[n]}{k}$ and $2 \le i \le n$, for any $F \in \mathcal{F}$, define the shift

$$S_i(F) = \begin{cases} F \setminus \{i\} \cup \{1\} & \text{if } i \in F \text{ and } 1 \notin F, \text{ and } F \setminus \{i\} \cup \{1\} \notin \mathcal{F}, \\ F & \text{otherwise.} \end{cases}$$

We write $S_i(\mathcal{F}) = \{S_i(F) : F \in \mathcal{F}\}$ for the resulting hypergraph after performing all shifting. We call $\mathcal{F}$ *compressed* if $S_i(\mathcal{F}) = \mathcal{F}$ for all $2 \le i \le n$.

Here is an example. Let $\mathcal{F} = \{134, 135, 234, 245\}$. Then $S_5(\mathcal{F}) = \{134, 135, 234, 124\}$. It is easy to see that the initial segment of colex ordering is compressed.

The following facts are easy to verified, we leave them as exercises. The first one ensures that during the shifting, the size of the hypergraph m remains the same.

Fact 2.5. *Let $\mathcal{F} \subseteq \binom{[n]}{k}$ and $2 \le i \le n$. The map $\mathcal{F} \to S_i(\mathcal{F})$ is injective. In particular,*

$$|\mathcal{F}| = |S_i(\mathcal{F})|.$$

The next one guarantees that we do not increase the size of shadow.

Fact 2.6. *Let $\mathcal{F} \subseteq \binom{[n]}{k}$ and $2 \le i \le n$. Then, $\partial(S_i(\mathcal{F})) \subseteq S_i(\partial\mathcal{F})$. In particular,*

$$|\partial(S_i(\mathcal{F}))| \le |S_i(\partial\mathcal{F})| = |\partial\mathcal{F}|.$$

For $\mathcal{F} \subseteq \binom{[n]}{k}$, we will write $\mathcal{F}_1 = \{F \in \mathcal{F} : 1 \in F\}$ and $\mathcal{F}_1^c = \mathcal{F} \setminus \mathcal{F}_1 = \{F \in \mathcal{F} : 1 \notin F\}$, and $\mathcal{L}_1 = \{F \setminus \{1\} : F \in \mathcal{F}_1\}$ for the link of 1.

Fact 2.7. *If $\mathcal{F}$ is compressed, then*

(i) $\partial\mathcal{F}_1^c \subseteq \mathcal{L}_1$; *and*

(ii) $\partial\mathcal{F} = \mathcal{L}_1 \cup \{E \cup \{1\} : E \in \partial\mathcal{L}_1\}$ *and so* $|\partial\mathcal{F}| = |\mathcal{L}_1| + |\partial\mathcal{L}_1|$.

2.4 Proof of Kruskal-Katona

We use double induction on the uniformity k and then on the size m.

The base case $k = 1$ is trivial as the shadow of any 1-uniform hypergraph is $\{\varnothing\}$.

Assume then $k \geq 2$ and induct now on $m = |\mathcal{F}|$. If $m = 1 = \binom{k}{k}$, then $\mathcal{F}$ has a single edge and the shadow is of size $\binom{k}{k-1}$ as desired.

Assume then $m \geq 2$. We may further assume that $\mathcal{F}$ is compressed, for otherwise, by Facts 2.5 and 2.6, we can keep performing shifting operation to $\mathcal{F}$ without changing the size or increasing the shadow.

Claim 2.8. $|\mathcal{L}_1| \geq \binom{n_k-1}{k-1} + \binom{n_{k-1}-1}{k-2} + \cdots + \binom{n_s-1}{s-1}$.

Proof. Suppose not, then

$$\begin{aligned}
|\mathcal{F}_1^c| &= |\mathcal{F}| - |\mathcal{F}_1| = |\mathcal{F}| - |\mathcal{L}_1| \\
&\geq \left(\binom{n_k}{k} + \binom{n_{k-1}}{k-1} + \cdots + \binom{n_s}{s}\right) - \left(\binom{n_k-1}{k-1} + \binom{n_{k-1}-1}{k-2} + \cdots + \binom{n_s-1}{s-1}\right) \\
&= \binom{n_k-1}{k} + \binom{n_{k-1}-1}{k-1} + \cdots + \binom{n_s-1}{s}.
\end{aligned}$$

Now, as $\mathcal{F}$ is compressed, $|\mathcal{F}_1^c| < |\mathcal{F}|$ and so applying induction hypothesis on $\mathcal{F}_1^c$ and using Fact 2.7(i), we get

$$|\mathcal{L}_1| \geq |\partial\mathcal{F}_1^c| > \binom{n_k-1}{k-1} + \binom{n_{k-1}-1}{k-2} + \cdots + \binom{n_s-1}{s-1},$$

a contradiction. $\square$

Finally, applying induction hypothesis on $\mathcal{L}_1$ and using Fact 2.7(ii), we get

$$\begin{aligned}
|\partial\mathcal{F}| &= |\mathcal{L}_1| + |\partial\mathcal{L}_1| = |\partial\mathcal{L}_1| \\
&\geq \left(\binom{n_k-1}{k-1} + \cdots + \binom{n_s-1}{s-1}\right) + \left(\binom{n_k-1}{k-2} + \cdots + \binom{n_s-1}{s-2}\right) \\
&= \binom{n_k-1}{k} + \cdots + \binom{n_s-1}{s}.
\end{aligned}$$

This completes the proof.