Lecture 10. Extremal set theory, part 2

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Today we start with the classical Erdős-Ko-Rado theorem and give a beautiful double-counting proof of it by Katona. Then we continue from last time to present two more applications of set-pairs inequalities, one on counting the maximal intersecting families and the other one on fractional Helly theorem.

1 Erdős-Ko-Rado and Katona's proof

Let us first state the classical Erdős-Ko-Rado theorem. A family $\mathcal{F} \subseteq 2^{[n]}$ of subsets of $[n]$ is *intersecting* if sets in $\mathcal{F}$ are pairwise intersecting, i.e. for any $F, F' \in \mathcal{F}$, $F \cap F' \neq \varnothing$. The question we are interested in here is how large can a k-uniform intersecting family be? We may assume that the ground set is of size at least $2k$, for otherwise every two k-sets would intersect.

Let us start by constructing a (large) uniform intersecting family. If we fix an element and take all k-sets containing this element, then the resulting family is certainly intersecting; we call such a family a *star*. Note that a star has size $\binom{n-1}{k-1}$. Can we do better? Erdős-Ko-Rado says "no".

Theorem 1.1 (Erdős-Ko-Rado)**.** *Let $n \geq 2k$. If a system $\mathcal{F}$ of k-sets in $[n]$ is intersecting, then*

$$|\mathcal{F}| \leq \binom{n-1}{k-1}.$$

Proof. Let C_n be the set of all cyclic permutation of $[n]$ and so $|C_n| = (n-1)!$. We shall double count the pairs (σ, F) of $\sigma \in C_n$ and $F \in \mathcal{F}$ such that F forms an interval in σ, i.e. elements of F appear consecutively in σ. Let P be the number of all such pairs, we shall see that

$$|\mathcal{F}| \cdot k! \cdot (n-k)! = P \leq (n-1)! \cdot k,$$

which would imply the desired upper bound on $|\mathcal{F}|$.

To see the first equality, we count the pairs from the point of view of $F \in \mathcal{F}$. For each $F \in \mathcal{F}$, as it has size k, it can occurs consecutively in $k!$ ways and the rest of the elements can be arranged in $(n-k)!$ ways, implying the equality above.

For the upper bound on P, it suffices to show that any cyclic permutation $\sigma \in C_n$ contains as an interval at most k sets from $\mathcal{F}$. To see this, fix a set $F \in \mathcal{F}$ that appears as an interval in σ, say with elements $x_1, \ldots, x_k$ in clockwise order. For each $i \in [k]$, denote by E_i the interval on σ ending at x_i, and by S_i the interval starting at x_i (so $S_1 = E_k = F$). As $\mathcal{F}$ is an intersecting family and $F \in \mathcal{F}$, all other sets contained as an interval in σ has to intersect F, and hence has to be one of the sets in $\{S_i, E_i\}_{i \in [k]}$. Note also that for each $1 \leq i \leq k-1$, $E_i \cap S_{i+1} = \varnothing$, and so $\mathcal{F}$ can contain at most one from each pair $\{E_i, S_{i+1}\}_{i \in [k-1]}$. Thus, σ contains at most k sets in $\mathcal{F}$ as intervals, finishing the proof. □

A more careful analysis of the proof yields the following.

Exercise 1.2. Prove that if $n \geq 2k+1$, then stars are the unique extremal examples.

Exercise 1.3. Give a non-star extremal example of k-uniform intersecting family when $n = 2k$.

What about intersecting families that are not necessarily uniform? A non-uniform star is still intersecting and has size 2^{n-1}. This turns out to be again optimal.

Exercise 1.4. If $\mathcal{F} \subseteq 2^{[n]}$ is intersecting, then $|\mathcal{F}| \leq 2^{n-1}$.

Exercise 1.5. Give a non-star example of intersecting family of size 2^{n-1}.

2 Counting maximal intersecting families

A recent application of the set-pair inequality is due to Balogh-Das-Delcourt-Liu-Sharifzadeh on the number of maximal intersecting families.

As every subfamily of a star is also intersecting, there are at least $2^{\binom{n-1}{k-1}}$ many k-uniform intersecting families. BDDLS determined the log-asymptotics of the number of uniform intersecting families.

Theorem 2.1. *For all $n \geq 2k+1$, the number of intersecting families on $[n]$ is*

$$2^{(1+o(1))\binom{n-1}{k-1}}.$$

We will not prove this statement here, but only remark that the bound on n here is optimal. Indeed, when $n = 2k$, there are exponentially more uniform intersecting families.

Exercise 2.2. When $n = 2k$, there are at least $3^{\binom{n-1}{k-1}}$ many k-uniform intersecting families.

On contrast, there are far fewer[1] maximal (under inclusion) intersecting families.

Theorem 2.3. *The number of maximal intersecting k-uniform families on $[n]$ is at most*

$$\sum_{i=0}^{\binom{2k}{k}} \binom{\binom{n}{k}}{i} \leq \binom{n}{k}^{\binom{2k}{k}}.$$

We need some notations for its proof. Given a family of sets $\mathcal{F}$, we denote by

$$\mathcal{I}(\mathcal{F}) = \left\{ G \in \binom{[n]}{k} : \forall F \in \mathcal{F},\ G \cap F \neq \emptyset \right\}$$

the family of all sets intersecting every set in $\mathcal{F}$. Note that $\mathcal{F}$ forms an intersecting family if and only if $\mathcal{F} \subset \mathcal{I}(\mathcal{F})$, while $\mathcal{F}$ is maximal if and only if $\mathcal{F} = \mathcal{I}(\mathcal{F})$. Given a maximal intersecting family, we call $\mathcal{G} \subset \mathcal{F}$ a *generating set* if $\mathcal{F} = \mathcal{I}(\mathcal{G})$.

The idea of the proof is to show that every maximal intersecting family admits a small generating set via set-pairs inequality, which then allows us to bound the number of maximal intersecting families.

Proof of Theorem 2.3. Let $\mathcal{F}_0 = \{F_1, F_2, \ldots, F_s\} \subseteq \mathcal{F}$ be a minimal generating set of $\mathcal{F}$. Observe that, by the minimality of $\mathcal{F}_0$, we have $\mathcal{F} \subsetneq \mathcal{I}(\mathcal{F}_0 \setminus \{F_i\})$ for each $1 \leq i \leq s$. Hence for each i we can find some set $G_i \in \mathcal{I}(\mathcal{F}_0 \setminus \{F_i\}) \setminus \mathcal{F}$. Since $G_i \in \mathcal{I}(\mathcal{F}_0 \setminus \{F_i\})$, we have $G_i \cap F_j \neq \varnothing$ for all $i \neq j$. Moreover, since $G_i \notin \mathcal{F} = \mathcal{I}(\mathcal{F}_0)$, we must further have $G_i \cap F_i = \varnothing$. Thus, we can apply Bollobás set-pairs inequality to $\{F_i\}$ and $\{G_i\}$ and get that $s \leq \binom{2k}{k}$. Therefore, $|\mathcal{F}_0| = s \leq \binom{2k}{k}$.

Map each maximal intersecting hypergraph $\mathcal{F}$ to a minimal generating set $\mathcal{F}_0 \subset \mathcal{F}$. As $\mathcal{F} = \mathcal{I}(\mathcal{F}_0)$, this map is injective. We have shown above that $|\mathcal{F}_0| \leq \binom{2k}{k}$, and hence the number of maximal intersecting hypergraphs is bounded by the number of sets of at most $\binom{2k}{k}$ edges, which is the sum above. □

[1] when $n \gg k$.

3 Fractionally Helly theorem

The final application is a simple proof, due to Alon-Kalai, of the fractionally Helly theorem. We will first lay out some backgrounds and prove it in the next lecture.

A collection $\mathcal{A} = \{A_1, \dots, A_k\}$ of convex sets in $\mathbb{R}^d$ is *pierced by a point*, if they have non-empty common intersection: $\cap\mathcal{A} := \cap_{i\in[k]} A_i \neq \varnothing$. Helly's theorem is a fundamental theorem in discrete geometry, deriving information of global intersection from information of local intersection.

Theorem 3.1 (Helly's theorem 1913)**.** *Let $\mathcal{F}$ be a collection of convex sets in $\mathbb{R}^d$. If every $(d+1)$-tuples $\mathcal{A} \in \binom{\mathcal{F}}{d+1}$ satisfy $\cap\mathcal{A} \neq \varnothing$, then $\cap\mathcal{F} \neq \varnothing$.*

There is a colourful version, which we state without proving it here.

Theorem 3.2 (Colourful Helly)**.** *Let $\mathcal{F}_1, \dots, \mathcal{F}_{d+1}$ be collections of convex sets in $\mathbb{R}^d$. If every rainbow $(d+1)$-tuples $\mathcal{A} = \{F_1, \dots, F_{d+1}\}$, $F_i \in \mathcal{F}_i$ for each $i \in [d+1]$, satisfy $\cap\mathcal{A} \neq \varnothing$, then there exists $i \in [d+1]$ such that $\cap\mathcal{F}_i \neq \varnothing$.*

The fractional Helly theorem says that if in a family of convex sets in $\mathbb{R}^d$, a positive fraction of $(d+1)$-tuples are each pierced by a point, then a positive fraction of all sets have non-empty common intersection.

Theorem 3.3 (Fractionally Helly theorem)**.** *Let $\alpha \in (0,1)$ and $\mathcal{F}$ be a collection of convex sets in $\mathbb{R}^d$. If at least $\alpha\binom{|\mathcal{F}|}{d+1}$ many $(d+1)$-tuples $\mathcal{A} \in \binom{\mathcal{F}}{d+1}$ satisfy $\cap\mathcal{A} \neq \varnothing$, then there exists a subcollection $\mathcal{F}' \subseteq \mathcal{F}$ such that $\cap\mathcal{F}' \neq \varnothing$ and $|\mathcal{F}'| \geq \beta|\mathcal{F}|$, where $\beta \geq (1 - (1-\alpha)^{\frac{1}{d+1}})$.*

Note that the bound above on β is optimal. Take, for instance, $d = 2$ and $\beta = (1-(1-\alpha)^{\frac{1}{3}})$, and let $\mathcal{F}$ consist of $\beta|\mathcal{F}|$ many copies of $\mathbb{R}^2$ and $(1-\beta)|\mathcal{F}|$ many lines in general position, i.e. no three cross a common point. Then every triple involving at most one line is pierced by a point, which is of $\frac{\binom{|\mathcal{F}|}{3} - \binom{(1-\beta)|\mathcal{F}|}{3}}{\binom{|\mathcal{F}|}{3}} = \alpha + o(1)$ fraction.