Lecture 1. Turán problem 1

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1 Turán theorem

Let us start with a fun puzzle. Suppose we can choose n irrational numbers $x_1, \dots, x_n$. How can we maximise the number of pairs (x_i, x_j) such that $x_i + x_j$ is rational? We shall see soon why this puzzle is relevant.

One of the most classical problems in extremal graph theory, nowadays so-called *Turán-type* problem, is:

Problem 1.1 (Turán-type)**.** How dense a graph can be without containing another graph as a subgraph?

More specifically, given a graph H, we say a graph G *contains a copy* of H, or H is a *subgraph* of G, denoted by $H \subseteq G$, if there is an injective map $\varphi : V(H) \to V(G)$ that preserves adjacencies, i.e. for any $uv \in E(H)$, we have $\varphi(u)\varphi(v) \in E(G)$. We call such a map an *embedding* of H in G. We say G is *H-free* if it does not contain H as a subgraph.

In Turán-type Problem 1.1, we study the *extremal number* of a graph H, defined as

$$\mathrm{ex}(n, H) = \max\{e(G) : |G| = n \text{ and } G \text{ is } H\text{-free}\},$$

i.e. the maximum size of an n-vertex H-free graph. We call an n-vertex graph G an *extremal graph* for H, if G is H-free of maximum size, i.e. $e(G) = \mathrm{ex}(n, H)$. One of the earliest applications of extremal graph theory, by Erdős, is to construct dense multiplicative Sidon set of integers using a graph without 4-cycles.

The first result in extremal graph theory is the following theorem of Mantel, which answers Problem 1.1 when forbidding triangles as subgraphs.

Theorem 1.2 (Mantel 1907)**.** *Let G be an n-vertex graph. If G is triangle-free, then*

$$e(G) \le \mathrm{ex}(n, K_3) = \lfloor n^2/4 \rfloor.$$

Exercise 1.3. Solve the puzzle at the beginning of this section, i.e. find the maximum number of pairs of irrationals (x_i, x_j) with $x_i + x_j$ being rational.

Exercise 1.4. Prove that for any k-vertex tree T, $\mathrm{ex}(n, T) \le kn$.

Mantel's result in fact shows that extremal graph for triangle is $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$. This answers also the following natural question for triangles.

Problem 1.5 (Extremal structure/Stability)**.** How do H-extremal graphs look like? What about almost extremal graphs[1], do they look like extremal ones?

Theorem 1.2 was later generalised by Turán to forbidding larger cliques. To state his result, we need to define a special family of graphs. Let $r \in \mathbb{N}$, the *r-partite Turán graph* on n vertices, denoted by $T_r(n)$, is the balanced complete r-partite n-vertex graph, i.e. each partite set is of size either $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$. Clearly, $T_r(n)$ is K_{r+1}-free.

[1]We say G is *almost* extremal for H if G is H-free and close to maximum size, i.e. $e(G) \ge ex(n, H) - o(n^2)$.

Theorem 1.6 (Turán 1941)**.** *Let $r \geq 2$ be an integer and G be an n-vertex graph. If G is K_{r+1}-free, then*

$$e(G) \leq \mathrm{ex}(n, K_{r+1}) = e(T_r(n)) = \left(1 - \frac{1}{r}\right)\frac{n^2}{2} - O(r).$$

Furthermore, the Turán graph $T_r(n)$ is the unique extremal graph.

We see from Turán theorem that there is a unique extremal graph $T_r(n)$. The following theorem of Erdős and Simonovits shows that this problem is stable in the sense that every almost extremal graph must be close in structure to the extremal Turán graph, answering Problem 1.5 for cliques.

Theorem 1.7 (Erdős-Simonovits stability 1966)**.** *Let $\varepsilon > 0$, there exists $\delta > 0$ such that the following holds. Let G be an n-vertex K_{r+1}-free graph. If*

$$e(G) \geq \mathrm{ex}(n, K_{r+1}) - \delta n^2,$$

then G can be changed to $T_r(n)$ by altering at most εn^2 adjacencies.

There are many proofs of Turán theorem, one of which is Zykov's symmetrisation. We shall present a proof of an asymptotic version using a variation of symmetrisation due to Motzkin and Straus. In particular, Theorem 1.7 follows from Theorem 2.1 below.

1.1 Notation

For a vector $\boldsymbol{v} \in \mathbb{R}^k$, its ℓ_p-norm is $\|\boldsymbol{v}\|_p = (\sum_{i \in [k]} v_i^p)^{1/p}$. For a k-by-k real symmetric matrix A, without further specification, we order its eigenvalues non-increasingly as $\lambda_1 \geq \ldots \geq \lambda_k$; its *Frobenius*, or *Hilbert–Schmidt*, *norm* is

$$\|A\|_F = \left(\sum_{i,j \in [k]} a_{i,j}^2\right)^{1/2} = \sqrt{\mathrm{tr}(A^2)} = \sqrt{\lambda_1^2 + \ldots + \lambda_k^2}.$$

When a graph G is given, we write A_G for its adjacency matrix.

2 Symmetrisation à la Motzkin-Straus

Motzkin and Straus gave a continuous version of Zykov's symmetrisation, implying an asymtotic version of Turán's theorem. To state their version, we need a couple of notations. For an n-vertex graph G and $\boldsymbol{x} \in \mathbb{R}^n$, define the quadratic form

$$f_G(\boldsymbol{x}) = \boldsymbol{x}^T A_G \boldsymbol{x} = \sum_{i,j \in [n]:\ v_i v_j \in E(G)} x_i x_j.$$

Write S_n for the simplex

$$S_n = \Big\{\boldsymbol{x} \in \mathbb{R}^n :\ \forall i \in [n], x_i \geq 0 \ \text{ and } \sum_{i \in [n]} x_i = 1\Big\}.$$

We can think of $\boldsymbol{x} \in S_n$ as a weight function on $V(G)$. Then each $\boldsymbol{x}$ corresponds naturally to a weighted subgraph $G_{\boldsymbol{x}}$ of G, and the quadratic form above records its size: $f_G(\boldsymbol{x}) = 2e(G_{\boldsymbol{x}})$. In particular, $f_G((\frac{1}{n}, \ldots, \frac{1}{n})^T) = \frac{2}{n^2} e(G)$.

Theorem 2.1 (Motzkin-Straus)**.** *Let G be an n-vertex graph with clique number $\omega(G) = k$, and $\boldsymbol{x} \in S_n$. Then there exists $\boldsymbol{y} \in S_n$ such that $f_G(\boldsymbol{y}) \geq f_G(\boldsymbol{x})$ and $\mathrm{supp}(\boldsymbol{y}) = K_k$. In particular,*

$$f_G(\boldsymbol{x}) \leq \frac{k-1}{k}, \quad \forall \boldsymbol{x} \in S_n.$$

The idea of the proof is 'mass transportation': if $\boldsymbol{x}$ has mass on two coordinates corresponding to a pair of non-adjacent vertices, then we can move the mass from one coordinate to another without decreasing $f_G(\cdot)$. This eventually leads to a vector whose support induces a clique.

Proof of Theorem 2.1. Take $\boldsymbol{y} \in S_n$ with minimal support such that $f_G(\boldsymbol{y}) \geq f_G(\boldsymbol{x})$. Suppose to the contrary that $\{v_1, v_2\} \subseteq \mathrm{supp}(\boldsymbol{y})$ is an independent set, then for any $\boldsymbol{z} = (z_1, z_2, 0, \ldots, 0)^T$, we have $\boldsymbol{z}^T A \boldsymbol{z} = 0$. Consequently, for any $\alpha \in \mathbb{R}$, writting $\boldsymbol{a} = 2\alpha A_G \boldsymbol{y}$, we have

$$f_G(\boldsymbol{y} + \alpha \boldsymbol{z}) = f_G(\boldsymbol{y}) + \boldsymbol{a}^T \cdot \boldsymbol{z}.$$

Now, setting $z_2 = -z_1$ and choosing appropriate z_1, we get $\boldsymbol{a}^T \cdot \boldsymbol{z} = (a_1 - a_2)z_1 \geq 0$. Then choosing appropriate α, we obtain $\boldsymbol{y'} = \boldsymbol{y} + \alpha \boldsymbol{z} \in S_n$ such that $f_G(\boldsymbol{y'}) \geq f_G(\boldsymbol{y})$ and $\mathrm{supp}(\boldsymbol{y'}) \subseteq \mathrm{supp}(\boldsymbol{y}) \setminus \{v_i\}$ for some $i \in [2]$, contradicting the choice of $\boldsymbol{y}$. □

Exercise 2.2. Prove the 'In particular' part of Theorem 2.1.

Füredi and Maleki recently extend the symmetrisation to multi-colour case.

Exercise 2.3. Let G be an n-vertex graph, $G_1, G_2 \subseteq G$ be its subgraphs, and $\boldsymbol{x} \in S_n$. Then there exists $\boldsymbol{y} \in S_n$ such that $f_{G_i}(\boldsymbol{y}) \geq f_{G_i}(\boldsymbol{x})$ for each $i \in [2]$ and $\alpha(G[\mathrm{supp}(\boldsymbol{y})]) \leq 2$.

2.1 A quick application

For a real symmetric matrix A, we can use its Frobenius norm to bound its largest eigenvalue. By definition, we have $\lambda_1^2 \leq \mathrm{tr}(A^2) = \|A\|_F^2$. This can be improved as follows. Note that the bound below is tight for cliques. Indeed, the adjacency matrix of K_k has $\lambda_1 = k-1$ and Frobenius norm $\sqrt{k^2 - k}$.

Exercise 2.4. Let G be a graph with clique number k and A be its adjacency matrix. Then

$$\lambda_1(A)^2 \leq \frac{k-1}{k} \|A\|_F^2.$$