A proof of Frankl's conjecture on cross-union families

Stijn Cambie* Jaehoon Kim† Hong Liu* Tuan Tran‡

Abstract

The families $\mathcal{F}_0, \ldots, \mathcal{F}_s$ of k-element subsets of $[n] := \{1, 2, \ldots, n\}$ are called *cross-union* if there is no choice of $F_0 \in \mathcal{F}_0, \ldots, F_s \in \mathcal{F}_s$ such that $F_0 \cup \ldots \cup F_s = [n]$. A natural generalization of the celebrated Erdős–Ko–Rado theorem, due to Frankl and Tokushige, states that for $n \leqslant (s+1)k$ the geometric mean of $|\mathcal{F}_i|$ is at most $\binom{n-1}{k}$. Frankl conjectured that the same should hold for the arithmetic mean under some mild conditions. We prove Frankl's conjecture in a strong form by showing that the unique (up to isomorphism) maximizer for the arithmetic mean of cross-union families is the natural one $\mathcal{F}_0 = \ldots = \mathcal{F}_s = \binom{[n-1]}{k}$.

1 Introduction

The most natural operations on sets are intersections and unions. These two seemingly simple operations provide surprisingly many exciting theory over collections of sets. The most famous such instances in extremal set theory are the theory on intersecting families and the theory of hypergraph matchings. The Erdős-Ko-Rado theorem is arguably the most foundational result in the former regime, while the Erdős matching conjecture is the most central theme in the latter. In this paper, we consider a problem of Frankl which has deep connections to both of these intricate theories.

Let us start by recalling the cornerstone Erdős-Ko-Rado theorem. We say a family $\mathcal{F}$ of sets is *intersecting* if $A \cap B \neq \varnothing$ for any $A, B \in \mathcal{F}$.

Theorem 1.1 ([6])**.** *Let n and k be two positive integers with $n \geqslant 2k$. Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ is an intersecting family, then $|\mathcal{F}| \leqslant \binom{n-1}{k-1}$.*

This concept of intersecting family further generalizes to $(s+1)$-*cross-intersecting* families, which is a collection $\mathcal{F}_0, \ldots, \mathcal{F}_s$ of families of sets where $\bigcap_{0 \leqslant i \leqslant s} A_i \neq \varnothing$ for any choice of $A_i \in \mathcal{F}_i$ for all $0 \leqslant i \leqslant s$. There have been numerous interesting generalizations of the Erdős-Ko-Rado theorem to cross-intersecting families. The maximum value of $\prod_{0 \leqslant i \leqslant s} |\mathcal{F}_i|$ was considered in, e.g. [21, 20, 15, 3] and the maximum value of $\sum_{0 \leqslant i \leqslant s} |\mathcal{F}_i|$ was considered in [16, 4].

In the case of $(s+1)$-cross intersecting families, if $n < (s+1)k/s$, then trivially $\mathcal{F}_0 = \cdots = \mathcal{F}_s = \binom{[n]}{k}$ provides the maximum possible collection. Otherwise, the most natural extremal example for maximum product is the collection with $\mathcal{F}_i = \{A \in \binom{[n]}{k} : 1 \in A\}$, and this indeed is the extremal example as shown in [15]. For the sum version, it is more delicate, as certain relations of s, k, n might yield a different maximum as in the case of e.g. [16, 4]. For a simple example, when $s = 1$ and $n > 2k \geqslant 4$, a very asymmetric collection $\mathcal{F}_0 = \{[k]\}$ and $\mathcal{F}_1 = \{A \in \binom{[n]}{k} : A \cap [k] \neq \varnothing\}$ provides a maximum sum. To better illustrate the relations between s, k, n, it is much more convenient to consider the complements of the sets rather than the sets itself.

*Mathematics Institute, University of Warwick, UK. E-mail: stijn.cambie@hotmail.com, h.liu.9@warwick.ac.uk. Supported by the UK Research and Innovation Future Leaders Fellowship MR/S016325/1.

†Department of Mathematical Sciences, KAIST, South Korea. E-mail: jaehoon.kim@kaist.ac.kr. Supported by the POSCO Science Fellowship of POSCO TJ Park Foundation.

‡Discrete Mathematics Group, Institute for Basic Science (IBS), South Korea. E-mail: tuantran@ibs.re.kr. Supported by the Institute for Basic Science (IBS-R029-Y1).

By considering complements of the sets in an $(s+1)$-cross-intersecting family, we obtain the following notion.

Definition 1.2. A collection $\mathcal{F}_0, \dots, \mathcal{F}_s$ of families of nonempty sets in $\binom{[n]}{k}$ is $(s+1)$-*cross-union* (or simply *cross-union*) if $\bigcup_{0 \leqslant i \leqslant s} A_i \neq [n]$ for any choice of $A_i \in \mathcal{F}_i$ for all $0 \leqslant i \leqslant s$.

Here, we only consider the case where the families are nonempty. With this definition, we are interested in values of (n,k,s) which ensures that $\mathcal{F}_0 = \dots = \mathcal{F}_s = \binom{[n-1]}{k}$ is (the unique up to isomorphism) cross-union collection maximizing the sum $\sum_{0 \leqslant i \leqslant s} |\mathcal{F}_i|$. Indeed, Frankl proposed the following conjecture in [12].

Conjecture 1.3 (Frankl, [12])**.** *Let $n = sk + \ell$ with $1 \leqslant \ell \leqslant k$ and $s \geqslant 2, k \geqslant 2$. Suppose that $\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_s \subset \binom{[n]}{k}$ are non-empty and cross-union. Then there exists $s_0 = s_0(\ell)$ such that the following holds for all $s \geqslant s_0$:*

$$\frac{|\mathcal{F}_0| + |\mathcal{F}_1| + \ldots + |\mathcal{F}_s|}{s+1} \leqslant \binom{n-1}{k}.$$

Here, the assumption that $n \leqslant (s+1)k$ is necessary as otherwise the union of $(s+1)$ sets of size k is never equal to $[n]$. On the other hand the assumption $n > sk$ is also very natural. Indeed, note that $\mathcal{F}_0, \dots, \mathcal{F}_{s+1}$ being cross-union implies that $\mathcal{F}_0, \dots, \mathcal{F}_s$ is also cross-union. Hence, assuming $\mathcal{F}_{s+1}$ being the smallest among the families $\mathcal{F}_0, \dots, \mathcal{F}_{s+1}$, we can always obtain that

$$\frac{|\mathcal{F}_0| + |\mathcal{F}_1| + \ldots + |\mathcal{F}_{s+1}|}{s+2} \leqslant \frac{|\mathcal{F}_0| + |\mathcal{F}_2| + \ldots + |\mathcal{F}_s|}{s+1}.$$

Hence, proving the above conjecture for $s = \lfloor \frac{n}{k} \rfloor$ yields the results on all larger values of s, hence the condition $sk < n \leqslant (s+1)k$ in Conjecture 1.3 is sensible.

We further remark that the condition $s \geqslant s_0(\ell)$ in Conjecture 1.3 is also necessary. Indeed, for small values of s, the conclusion of Conjecture 1.3 does not always hold. For example, Hilton-Milner [16] proved that for $s = 1$, the maximum of $\frac{1}{s+1} \sum_{0 \leqslant i \leqslant s} |\mathcal{F}_i|$ is not $\binom{n-1}{k}$. Moreover, the following example shows that the value s_0 must depend on ℓ.

Example 1.4. For $k = \ell + c$, $s \geqslant 2, c \geqslant 1$, the families $\mathcal{F}_0 = \{[k]\}$, $\mathcal{F}_1 = \{A \in \binom{[n]}{k} : |A \cap [k]| \geqslant c+1\}$ and $\mathcal{F}_2 = \dots = \mathcal{F}_s = \binom{[n]}{k}$ are cross-union.

In fact, this example shows that $s_0 = \Omega(\frac{\ell}{\log \ell})$ is necessary. For fixed c and s, we know that $\binom{k}{\leqslant c} \leqslant (c+1)k^c$ and $\binom{n-1}{k} = \frac{n-k}{n}\binom{n}{k}$. If $s < \frac{k}{(c+2)\log k} - 1$, where $k \geqslant 3$, then $\frac{\binom{n-k}{k}}{\binom{n}{k}} \leqslant \left(\frac{n-k}{n}\right)^k \leqslant \left(\frac{s}{s+1}\right)^k < e^{-k/(s+1)} \leqslant \frac{1}{k^{c+2}} \leqslant \frac{1}{(c+2)nk^c}$. Hence, in this case, Example 1.4 satisfies

$$\begin{aligned}\sum_{0 \leqslant i \leqslant s} |\mathcal{F}_i| &\geqslant 1 + s\binom{n}{k} - \binom{k}{\leqslant c}\binom{n-k}{k} \geqslant s\binom{n}{k} - (c+1)k^c\left(\frac{s}{s+1}\right)^k\binom{n}{k} \\ &> \left(s - \frac{c}{n}\right)\binom{n}{k} = (s+1)\frac{n-k}{n}\binom{n}{k} = (s+1)\binom{n-1}{k}.\end{aligned}$$

Towards Conjecture 1.3, Frankl [11] proved sporadic cases. The main result in this paper is the following theorem verifying a strong form of Conjecture 1.3, yielding the uniqueness of the extremal example.

Theorem 1.5. *Let $n = sk + \ell$ with $1 \leqslant \ell \leqslant k$ and $s \geqslant 4\ell$. Suppose that $\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_s \subset \binom{[n]}{k}$ are non-empty and cross-union. Then*

$$\frac{|\mathcal{F}_0| + |\mathcal{F}_1| + \ldots + |\mathcal{F}_s|}{s+1} \leqslant \binom{n-1}{k}.$$

Furthermore, equality is attained only if $\mathcal{F}_0 = \ldots = \mathcal{F}_s = \binom{[n]\setminus\{i\}}{k}$ for some $i \in [n]$.

In view of Example 1.4, the linear bound $s \geqslant 4\ell$ above is best possible up to a logarithmic factor.

We remark that Conjecture 1.3 has a clear connection with the Erdős matching conjecture. A collection of s sets in $[n]$ is a *matching of size* s if they are pairwise disjoint.

Conjecture 1.6 (The Erdős matching conjecture)**.** *Let $n \geqslant k(s+1)$ and $\mathcal{F} \subset \binom{[n]}{k}$ has no matching of size $s+1$. Then*

$$|\mathcal{F}| \leqslant \max\left\{\binom{n}{k} - \binom{n-s}{k}, \binom{k(s+1)-1}{k}\right\}.$$

The Erdős matching conjecture is known to be true for n sufficiently large in terms of s and k since the publication of Erdős work [5], as well as in a small range where $n = k(s+1) + \ell$ for some $\ell = \varepsilon(k)(s+1)$ by Frankl [10]. There has been many interesting works [2, 17, 9, 13] improving the bound on n where this conjecture holds.

In the case of $n = k(s+1)$ and $\mathcal{F}_0 = \cdots = \mathcal{F}_s = \mathcal{F}$, this family being cross-union is the same as $\mathcal{F}$ having no matching of size $s+1$. From this, one can naturally consider several 'cross' versions of the Erdős matching conjecture. We will discuss some variants of the 'cross' version of the Erdős matching conjecture at the concluding remarks.

2 Preliminaries

For a set family $\mathcal{F}$, the shadow of $\mathcal{F}$ at level s is defined by

$$\sigma_s(\mathcal{F}) = \{G\colon |G| = s, \exists F \in \mathcal{F} \text{ with } G \subset F\}.$$

The following theorem by Frankl will be useful for us. A family $\mathcal{F}$ is *r-wise union* if $\bigcup_{0\leqslant i\leqslant s} A_i \neq [n]$ for any choice of $A_i \in \mathcal{F}$ for all $0 \leqslant i \leqslant s$.

Theorem 2.1 ([7])**.** *Let n, k and r be positive integers with $r \geqslant 2$ and $n \leqslant rk$. Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ is an r-wise union family. Then $|\mathcal{F}| \leqslant \binom{n-1}{k}$.*

2.1 Combinatorial lemmas

In this section we collect several combinatorial results that are needed for the proof of Theorem 1.5. A basic result of Frankl [8] (see Lemma 2.2 below) allows us to restrict ourself to *shifted* families. We say that a family $\mathcal{F} \subset \binom{[n]}{k}$ is shifted if for any $F = \{x_1, \ldots, x_k\} \in \mathcal{F}$ and any $G = \{y_1, \ldots, y_k\} \subset [n]$ such that $y_i \leqslant x_i$ for every $1 \leqslant i \leqslant k$, we have $G \in \mathcal{F}$. It is easy to see that if $\mathcal{F} \subset \binom{[n]}{k}$ is non-empty and shifted, then $[k] \in \mathcal{F}$.

Lemma 2.2. *Suppose that the families $\mathcal{F}_0, \ldots, \mathcal{F}_s \subset \binom{[n]}{k}$ are cross-union. Then there exist shifted and cross-union families $\mathcal{F}'_0, \ldots, \mathcal{F}'_s \subset \binom{[n]}{k}$ such that $|\mathcal{F}_i| = |\mathcal{F}'_i|$ for $0 \leqslant i \leqslant s$.*

The second lemma is another result of Frankl [12, Lemma 2.4], which is a probabilistic version of Katona's circle method. We give its proof for completeness.

Lemma 2.3. *Let $k_0, k_1, \ldots, k_s, n$ be positive integers with $k_0 + k_1 + \ldots + k_s \geqslant n$. Suppose that the families $\mathcal{G}_i \subset \binom{[n]}{k_i}$, $0 \leqslant i \leqslant s$, are cross-union. Then*

$$\sum_{i=0}^{s} \frac{|\mathcal{G}_i|}{\binom{n}{k_i}} \leqslant s.$$

Proof. Fix $s+1$ sets $A_0, \ldots, A_s$ satisfying $|A_0| = k_0, \ldots, |A_s| = k_s$ and $A_0 \cup \ldots \cup A_s = [n]$. Choose a permutation σ of $[n]$ uniformly at random. Let X denote the number of indices i with $\sigma(A_i) \in \mathcal{G}_i$. Since $\mathbb{P}(\sigma(A_i) \in \mathcal{G}_i) = |\mathcal{G}_i|/\binom{n}{k_i}$, we have $\mathbb{E}[X] = \sum_{i=0}^{s}|\mathcal{G}_i|/\binom{n}{k_i}$, by the linearity of expectation. On the other hand, the cross-union property implies $X \leqslant s$, resulting in $\mathbb{E}[X] \leqslant s$. Therefore, $\sum_{i=0}^{s}|\mathcal{G}_i|/\binom{n}{k_i} \leqslant s$. $\square$

We will require the following slightly weaker version of the Kruskal–Katona theorem, due to Lovász [19].

Theorem 2.4. *If $\mathcal{F} \subset \binom{[n]}{k}$ and $|\mathcal{F}| = \binom{x}{k}$ with $x \geqslant k$, then $|\sigma_{k-1}(\mathcal{F})| \geqslant \binom{x}{k-1}$.*

2.2 Technical lemmas

The following two technical lemmas will be used.

Lemma 2.5. *Let $n = ks + \ell$ with $1 \leqslant \ell \leqslant k$ and $s \geqslant 4\ell$. The following holds.*

(i) *If $k \geqslant 2\ell$, then $(s+1)\binom{n-1}{k} - s\binom{n}{k} + \binom{ks}{k} \geqslant \frac{\ell}{k}\binom{n}{k}$.*

(ii) *If $k < 2\ell$, then $(s+1)\binom{n-1}{k} - s\binom{n}{k} + \binom{ks}{k} \geqslant \binom{(1-1/k)n+1}{k}$.*

Proof. (i) For $x \geqslant ks$, we have

$$\binom{x-1}{k} = \frac{x-k}{x}\binom{x}{k} \geqslant \left(1 - \frac{1}{s}\right)\binom{x}{k}.$$

By iterating this and noting that $n - ks = \ell$, we obtain $\binom{ks}{k} \geqslant \left(1 - \frac{1}{s}\right)^{\ell}\binom{n}{k} \geqslant \left(1 - \frac{\ell}{s}\right)\binom{n}{k}$. Since $n \geqslant ks$, we obtain $\binom{n-1}{k} \geqslant \left(1 - \frac{1}{s}\right)\binom{n}{k}$. Therefore,

$$(s+1)\binom{n-1}{k} - s\binom{n}{k} + \binom{ks}{k} \geqslant \left(1 - \frac{\ell+1}{s}\right)\binom{n}{k} \geqslant \frac{\ell}{k}\binom{n}{k}$$

assuming $k \geqslant 2\ell$ and $s \geqslant 4\ell$.

(ii) Since $2\ell \geqslant k+1$ and $s \geqslant 4\ell$, we have $n \geqslant ks \geqslant 2k^2 + 2k$. Thus

$$\frac{\binom{n-1}{k-1}}{\binom{n-2k}{k-1}} \leqslant \left(\frac{n-k+1}{n-3k+2}\right)^{k-1} \leqslant \left(1 + \frac{1}{k}\right)^{k-1} \leqslant k.$$

It follows that

$$\begin{aligned}
(s+1)\binom{n-1}{k} - s\binom{n}{k} + \binom{ks}{k} &\geqslant \binom{ks}{k} - \binom{n-1}{k-1} \\
&\geqslant \binom{n-k}{k} - k\binom{n-2k}{k-1} \\
&\geqslant \binom{n-2k}{k} \\
&\geqslant \binom{(1-1/k)n+1}{k},
\end{aligned}$$

where in the first line we used $(s+1)\binom{n-1}{k} = \left(s - \frac{k-\ell}{n}\right)\binom{n}{k} = s\binom{n}{k} - \binom{n-1}{k-1} + \frac{\ell}{n}\binom{n}{k}$, the third inequality holds since $\binom{n-k}{k} - \binom{n-2k}{k} = \sum_{m=n-2k}^{n-k-1}\binom{m}{k-1} \geqslant k\binom{n-2k}{k-1}$, and in the last inequality we used $n \geqslant 2k^2 + 2k$. $\square$

Lemma 2.6. *Let k, ℓ, n be integers with $1 \leqslant \ell \leqslant k < n$. Let $x_0 \in [k, n-1]$ be a real number for which*

$$\frac{\binom{x_0}{\ell}}{\binom{n}{\ell}} \leqslant \frac{k}{\ell}\frac{\binom{x_0}{k}}{\binom{n}{k}}. \tag{1}$$

Then

$$\frac{\binom{x_0}{\ell}}{\binom{n}{\ell}} \geqslant \frac{\binom{x_0}{k}}{\binom{n}{k}} + \frac{k-\ell}{n}.$$

Proof. We write $A(x) = \frac{\binom{x}{k}}{\binom{n}{k}}$ and $B(x) = \frac{\binom{x}{\ell}}{\binom{n}{\ell}}$. Consider the function $f(x) = B(x) - A(x)$, where $x_0 \leqslant x \leqslant n-1$. We wish to show $f(x_0) \geqslant f(n-1) = \frac{k-\ell}{n}$.

Notice first that

$$f'(x) = B(x)\left(\frac{1}{x} + \frac{1}{x-1} + \ldots + \frac{1}{x-\ell+1}\right) - A(x)\left(\frac{1}{x} + \frac{1}{x-1} + \ldots + \frac{1}{x-k+1}\right). \tag{2}$$

By the assumption, $\frac{A(x_0)}{B(x_0)} \geqslant \frac{\ell}{k}$. Hence

$$\begin{aligned}\frac{A(x)}{B(x)} = \prod_{i=\ell}^{k-1}\frac{x-i}{n-i} &\geqslant \prod_{i=\ell}^{k-1}\frac{x_0-i}{n-i}\\ &= \frac{A(x_0)}{B(x_0)} \geqslant \frac{\ell}{k}.\end{aligned} \tag{3}$$

As $\frac{1}{x} \leqslant \frac{1}{x-1} \leqslant \ldots \leqslant \frac{1}{x-\ell+1} \leqslant \ldots \leqslant \frac{1}{x-k+1}$, we see that

$$\frac{1}{x} + \frac{1}{x-1} + \ldots + \frac{1}{x-k+1} \geqslant \frac{k}{\ell}\left(\frac{1}{x} + \frac{1}{x-1} + \ldots + \frac{1}{x-\ell+1}\right). \tag{4}$$

From (2), (3) and (4) we conclude $f'(x) \leqslant 0$ for every $x \in [x_0, n-1]$. Thus $f(x_0) \geqslant f(n-1) = \frac{k-\ell}{n}$, as desired. □

3 Proof of Frankl's conjecture

We are now ready to prove Theorem 1.5.

Proof of Theorem 1.5. Suppose to the contrary that $\sum_{i=0}^{s}|\mathcal{F}_i| > (s+1)\binom{n-1}{k}$, or equivalently,

$$\sum_{i=0}^{s}\frac{|\mathcal{F}_i|}{\binom{n}{k}} > \frac{(s+1)\binom{n-1}{k}}{\binom{n}{k}} = s - \frac{k-\ell}{n}. \tag{5}$$

Claim 3.1. We can assume $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_s$.

Proof of claim. From Lemma 2.2 we can assume the families $\mathcal{F}_i$'s are non-empty and shifted. In particular, $[k] \in \mathcal{F}_i$ for every $0 \leqslant i \leqslant s$. For a fixed pair $1 \leqslant u < v \leqslant n$, replacing $\mathcal{F}_u$ and $\mathcal{F}_v$ by $\mathcal{F}_u \cap \mathcal{F}_v$ and $\mathcal{F}_u \cup \mathcal{F}_v$ will preserve the nonemptiness, the cross-union property, and the sum $\sum_{i=0}^{s}|\mathcal{F}_i|$. Iterating this operation for all pairs $1 \leqslant u < v \leqslant n$ will generate families $\mathcal{F}_0, \ldots, \mathcal{F}_s$ with desired nested properties. ■

Since $\mathcal{F}_0 \subset \mathcal{F}_i$ for $1 \leqslant i \leqslant s$, $\mathcal{F}_0$ is $(s+1)$-wise union. By Theorem 2.1, one must have $|\mathcal{F}_0| \leqslant \binom{n-1}{k}$. So we can write $|\mathcal{F}_0| = \binom{x_0}{k}$ for some $k \leqslant x_0 \leqslant n-1$.

Since the families $\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_s$ are non-empty and cross-union, so are the families $\sigma_\ell(\mathcal{F}_0), \mathcal{F}_1, \ldots, \mathcal{F}_s$. Thus Lemma 2.3 applies. We conclude

$$\frac{|\sigma_\ell(\mathcal{F}_0)|}{\binom{n}{\ell}} + \sum_{i=1}^{s} \frac{|\mathcal{F}_i|}{\binom{n}{k}} \leqslant s. \tag{6}$$

Furthermore, as $|\mathcal{F}_0| = \binom{x_0}{k}$ with $x_0 \geqslant k$, Theorem 2.4 implies

$$|\sigma_\ell(\mathcal{F}_0)| \geqslant \binom{x_0}{\ell}. \tag{7}$$

From (5), (6) and (7), we see that

$$\frac{\binom{x_0}{\ell}}{\binom{n}{\ell}} < \frac{|\mathcal{F}_0|}{\binom{n}{k}} + \frac{k-\ell}{n} = \frac{\binom{x_0}{k}}{\binom{n}{k}} + \frac{k-\ell}{n}. \tag{8}$$

Hence, in order to get a contradiction, it suffices to show that x_0 satisfies the conditions of Lemma 2.6. As an intermediary step, we bound the size of $\mathcal{F}_0$ from below.

Claim 3.2. $|\mathcal{F}_0| > (s+1)\binom{n-1}{k} - s\binom{n}{k} + \binom{ks}{k}$.

Proof of claim. As $\mathcal{F}_0$ is non-empty, it contains some $F_0 \in \binom{[n]}{k}$. Fix an arbitrary subset $X \subset [n]$ satisfying $|X| = ks$ and $F_0 \cup X = [n]$. For $1 \leqslant i \leqslant s$, define $\mathcal{G}_i = \mathcal{F}_i \cap \binom{X}{k}$. Notice that the families $\mathcal{G}_1, \ldots, \mathcal{G}_s$ are cross-union. Indeed, if $G_i \in \mathcal{G}_i$, $1 \leqslant i \leqslant s$, satisfy $G_1 \cup \ldots \cup G_s = X$, then adding $F_0 \in \mathcal{F}_0$ we get a contradiction to the cross-union property of $\mathcal{F}_0, \ldots, \mathcal{F}_s$.

Applying Lemma 2.3 to s families $\mathcal{G}_1, \ldots, \mathcal{G}_s \subset \binom{X}{k}$ yields $\sum_{i=1}^{s} |\mathcal{G}_i| \leqslant (s-1)\binom{ks}{k}$. So

$$\sum_{i=1}^{s} |\mathcal{F}_i| \leqslant \sum_{i=1}^{s} \left(|\mathcal{G}_i| + \binom{n}{k} - \binom{ks}{k} \right) \leqslant s\binom{n}{k} - \binom{ks}{k}.$$

Together with (5) this entails $|\mathcal{F}_0| > (s+1)\binom{n-1}{k} - s\binom{n}{k} + \binom{ks}{k}$, as desired. ∎

Claim 3.3. x_0 meets the conditions of Lemma 2.6. In particular, x_0 does not satisfy (8).

Proof of claim. We know that $k \leqslant x_0 \leqslant n-1$. It remains to show $\frac{\binom{x_0}{\ell}}{\binom{n}{\ell}} \leqslant \frac{k}{\ell} \frac{\binom{x_0}{k}}{\binom{n}{k}}$. In order to do this, we distinguish two cases.

Case 1: $k \geqslant 2\ell$**.** It follows from Claim 3.2 and Lemma 2.5 (i) that $\frac{\binom{x_0}{k}}{\binom{n}{k}} \geqslant \frac{\ell}{k}$. Moreover, $\frac{\binom{x_0}{\ell}}{\binom{n}{\ell}} \leqslant 1$ for $x_0 \leqslant n$. Hence $\frac{\binom{x_0}{\ell}}{\binom{n}{\ell}} \leqslant \frac{k}{\ell} \frac{\binom{x_0}{k}}{\binom{n}{k}}$.

Case 2: $k < 2\ell$**.**

From Claim 3.2 and Lemma 2.5 (ii), we get $x_0 \geqslant (1 - 1/k)n + 1$. Hence

$$\begin{aligned}
\frac{\binom{x_0}{k}}{\binom{n}{k}} &= \frac{\binom{x_0}{\ell}}{\binom{n}{\ell}} \cdot \prod_{i=\ell}^{k-1} \frac{x_0 - i}{n - i} \\
&\geqslant \frac{\binom{x_0}{\ell}}{\binom{n}{\ell}} \cdot \left(\frac{x_0 - k}{n - k} \right)^{k-\ell} \\
&\geqslant \frac{\binom{x_0}{\ell}}{\binom{n}{\ell}} \cdot \left(1 - \frac{1}{k} \right)^{k-\ell} \\
&\geqslant \frac{\binom{x_0}{\ell}}{\binom{n}{\ell}} \cdot \frac{\ell}{k},
\end{aligned}$$

as required. $\blacksquare$

This completes our proof of Theorem 1.5. $\square$

4 Concluding remarks

One remaining question is to determine the smallest value of s_0 where Conjecture 1.3 holds. As our theorem provides that this best value of s_0 is at most 4ℓ while the example at the introduction shows that it must be $\Omega(\frac{\ell}{\log \ell})$. It would be interesting to determine the correct order of magnitude for $s_0(\ell)$.

Another interesting question is what happens when s is smaller than the above value $s_0(\ell)$. In such a case, would Example 1.4 provide an extremal example? In particular, would the answer of the following question be true?

Question 4.1. *Let $n = ks + \ell$ where $0 < \ell < k$ and $\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_s \subset \binom{[n]}{k}$ be non-empty cross-union families. Does the following inequality hold?*

$$\sum_{i=0}^{s} |\mathcal{F}_i| \leqslant \max\left\{(s+1)\binom{n-1}{k}, 1 + s\binom{n}{k} - \sum_{i=0}^{k-\ell}\binom{k}{i}\binom{n-k}{k-i}\right\}$$

On the other hand, Conjecture 1.3 motivates the 'cross' version of the Erdős matching conjecture as follows.

In [14], Frankl and Kupavskii defined that families $\mathcal{F}_0, \dots, \mathcal{F}_s$ satisfies the property $U(s+1, q)$ if $|F_0 \cup F_1 \cup \dots \cup F_s| \leqslant q$ for every choice of $F_i \in \mathcal{F}_i$, $0 \leqslant i \leqslant s$. The condition of being cross-union is the same as having the property $U(s+1, n-1)$ and the condition on the Erdős matching conjecture is the same as $\mathcal{F}_0 = \dots = \mathcal{F}_{s+1} = \mathcal{F}$ having the property $U(s+1, k(s+1)-1)$. This provides the natural 'cross' version of the Erdős matching conjecture by considering the geometric mean and arithmetic mean of families satisfying the condition $U(s+1, k(s+1)-1)$.

For the maximum value of $\prod_{0 \leqslant i \leqslant s} |\mathcal{F}_i|$ where $\mathcal{F}_0, \dots, \mathcal{F}_s$ having property $U(s+1, k(s+1)-1)$, one can naturally consider $\mathcal{F}_0 = \mathcal{F}_1 = \{A \in \binom{[n]}{k} : 1 \in A\}$ and $\mathcal{F}_2 = \dots = \mathcal{F}_s = \binom{[n]}{k}$. In fact, the following proposition provides that this is an extremal example provided that n is sufficiently large.

Proposition 4.2. *For $k, s \geqslant 1$, there exists $n_0(k, s)$ such that the following holds for all $n \geqslant n_0(k, s)$. Suppose that $\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_s \subset \binom{[n]}{k}$ are non-empty families having the property $U(s+1, k(s+1)-1)$. Then we have*

$$\prod_{i=0}^{s} |\mathcal{F}_i| \leqslant \binom{n-1}{k-1}^2 \binom{n}{k}^{s-1}.$$

Here the $s = 1$ case is known by e.g. the result of Pyber [21] and for $s \geqslant 2$ it is sufficient to note that for n sufficiently large, $\left(\binom{n}{k} - \binom{n-ks}{k}\right)^{s+1}$ is smaller than the expression in the proposition. If $\mathcal{F}_s$ is the largest family and the other s families have k pairwise disjoint sets, then all families have size at most $|\mathcal{F}_s| \leqslant \binom{n}{k} - \binom{n-ks}{k}$ as desired. If this is not the case, then the result follows by induction on s.

On the other hand, it is interesting whether the above bound is actually best possible when n is close to ks. For all we know, $\binom{n-1}{k-1}^{s+1}$ can be the correct maximum when n is just above ks.

For the maximum value of $\sum_{0 \leqslant i \leqslant s} |\mathcal{F}_i|$, the families $\mathcal{F}_0 = [k], \mathcal{F}_1 = \{A \in \binom{[n]}{k} : |A \cap [k]| \geqslant 1\}$ and $\mathcal{F}_2 = \dots = \mathcal{F}_s = \binom{[n]}{k}$ are natural candidates for an extremal example. The following proposition yields that indeed this is an extremal example for sufficiently large n.

Proposition 4.3. *For $k, s \geqslant 1$, there exists $n_0(k,s)$ such that the following holds for all $n \geqslant n_0(k,s)$. Suppose that $\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_s \subset \binom{[n]}{k}$ are non-empty families having the property $U(s+1, k(s+1)-1)$. Then we have*

$$\sum_{i=0}^{s} |\mathcal{F}_i| \leqslant 1 + s\binom{n}{k} - \binom{n-k}{k}.$$

Here the case of $s = 1$ is known by the result of Hilton and Milner [16] and a similar induction as before works as $(s+1)\left(\binom{n}{k} - \binom{n-ks}{k}\right)$ is smaller than the expression in the proposition for n sufficiently large.

Even when $n = ks + \ell$ with small ℓ, as long as $k > \ell$, the term $1 + s\binom{n}{k} - \binom{n-k}{k}$ is bigger than $(s+1)\binom{ks-1}{k}$. Hence, the above example shows that, unlike Conjecture 1.3, $\mathcal{F}_0 = \dots = \mathcal{F}_s = \binom{[ks-1]}{k}$ is not an extremal example when $n > k(s+1)$.

While the maximum of the geometric mean and the arithmetic mean of the families satisfying $U(s+1, k(s+1)-1)$ may behave differently from what is conjectured in the Erdős matching conjecture, it has been conjectured [1, 17] that the minimum size behaves as in the Erdős matching conjecture.

Conjecture 4.4 ([1, 17])**.** *Let $\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_s \subset \binom{[n]}{k}$ be non-empty families, where $n \geqslant k(s+1)$, such that for any choice of $F_i \in \mathcal{F}_i$, $0 \leqslant i \leqslant s$ one has $|F_0 \cup F_1 \cup \ldots \cup F_s| \leqslant k(s+1)-1$. Then*

$$\min\{|\mathcal{F}_0|, |\mathcal{F}_1|, \dots, |\mathcal{F}_s|\} \leqslant \max\left\{\binom{n}{k} - \binom{n-s}{k}, \binom{k(s+1)-1}{k}\right\}.$$

Recently Kupavskii [18] proved that this conjecture when s is sufficiently large and $n > 3e(s+1)k$.

References

[1] R. Aharoni and D. Howard. Size conditions for the existence of rainbow matchings. *preprint*, 1.

[2] B. Bollobás, D. E. Daykin, and P. Erdős. Sets of independent edges of a hypergraph. *Quart. J. Math. Oxford Ser. (2)*, 27(105):25–32, 1976.

[3] P. Borg. The maximum product of sizes of cross-intersecting families. *Discrete Math.*, 340(9):2307–2317, 2017.

[4] P. Borg and C. Feghali. The maximum sum of sizes of cross-intersecting families of subsets of a set. *arXiv e-prints*, page arXiv:2012.15356, Dec. 2020.

[5] P. Erdős. A problem on independent r-tuples. *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.*, 8:93–95, 1965.

[6] P. Erdős, C. Ko, and R. Rado. Intersection theorems for systems of finite sets. *Quart. J. Math. Oxford Ser. (2)*, 12:313–320, 1961.

[7] P. Frankl. On Sperner families satisfying an additional condition. *J. Combinatorial Theory Ser. A*, 20(1):1–11, 1976.

[8] P. Frankl. The shifting technique in extremal set theory. In *Surveys in combinatorics 1987 (New Cross, 1987)*, volume 123 of *London Math. Soc. Lecture Note Ser.*, pages 81–110. Cambridge Univ. Press, Cambridge, 1987.

[9] P. Frankl. Improved bounds for Erdős' matching conjecture. *J. Combin. Theory Ser. A*, 120(5):1068–1072, 2013.

[10] P. Frankl. Proof of the Erdős matching conjecture in a new range. *Israel J. Math.*, 222(1):421–430, 2017.

[11] P. Frankl. Old and new applications of Katona's circle. *European J. Combin.*, 95:Paper No. 103339, 21, 2021.

[12] P. Frankl. On the arithmetic mean of the size of cross-union families. *Acta Math. Hungar.*, 164(1):312–325, 2021.

[13] P. Frankl and A. Kupavskii. The Erdős Matching Conjecture and concentration inequalities. *arXiv e-prints*, page arXiv:1806.08855, June 2018.

[14] P. Frankl and A. Kupavskii. Beyond the Erdős matching conjecture. *European J. Combin.*, 95:Paper No. 103338, 12, 2021.

[15] P. Frankl and N. Tokushige. On r-cross intersecting families of sets. *Combin. Probab. Comput.*, 20(5):749–752, 2011.

[16] A. J. W. Hilton and E. C. Milner. Some intersection theorems for systems of finite sets. *Quart. J. Math. Oxford Ser. (2)*, 18:369–384, 1967.

[17] H. Huang, P.-S. Loh, and B. Sudakov. The size of a hypergraph and its matching number. *Combin. Probab. Comput.*, 21(3):442–450, 2012.

[18] A. Kupavskii. Rainbow version of the Erdős Matching Conjecture via Concentration. *arXiv e-prints*, page arXiv:2104.08083, Apr. 2021.

[19] L. Lovász. *Combinatorial problems and exercises*. North-Holland Publishing Co., Amsterdam-New York, 1979.

[20] M. Matsumoto and N. Tokushige. The exact bound in the Erdős-Ko-Rado theorem for cross-intersecting families. *J. Combin. Theory Ser. A*, 52(1):90–97, 1989.

[21] L. Pyber. A new generalization of the Erdős-Ko-Rado theorem. *J. Combin. Theory Ser. A*, 43(1):85–90, 1986.