Cycles and trees in graphs

Hong Liu

22nd October 2020

Abstract

Techniques utilising pseudorandomness and expansions have become standard in extremal combinatorics. Two meta-questions can be asked here.

1. What condition on a graph G guarantees that G or some subgraph of it has certain pseudorandomness or expansion property?
2. What substructure we can force in G, assuming that G has some pseudorandomness or expansion property?

In this note, we will go around the 2nd topic and cover some techniques on embedding cycles/trees in graphs with pseudorandom/expansion properties.

Contents

1 Introduction **1**

2 Preliminaries **2**

2.1 Graph notions 2

2.2 Erdős-Rényi binomial random graph 2

3 Depth First Search **3**

3.1 Long paths in $G(n, p)$, part 1 4

3.2 Long cycles in expander, part 1 5

4 Pósa's rotation and extension, and edge sprinkling **5**

4.1 Long paths in $G(n, p)$, part 2 6

4.2 Long cycles in expander, part 2 7

4.3 Extension and edge sprinkling 7

4.4 Finding large $(2, n/5)$-expander in pseudorandom graphs 8

4.5 Long cycles and Hamiltonicity in $G(n, p)$, part 1 9

4.6 Long cycles and Hamiltonicity in $G(n, p)$, part 2 10

5 Friedman-Pippenger theorem on tree embedding **11**

5.1 Extendability method 12

5.2 Large trees in expanders with large set connectivity 13

5.3 Almost spanning bounded degree trees in $G(n, p)$ 14

6 Spanning bounded degree trees in $G(n, p)$ **14**

6.1 Covering expanders with paths 15

6.2 Linking with specified ends and lengths 16

6.3 Absorber from expansion 17

6.4 Proof sketch of Theorem 6.7 18

1 Introduction

Cycles and trees are among the most basic graphs, the embedding problems of which have a long history. As mentioned in the abstract, we will cover a couple of methods on embedding cycles/trees in (pseudo)random graphs. If you spot typos or have suggestions for better presentation, please email me. :)

2 Preliminaries

For $n \in \mathbb{N}$, let $[n] := \{1, \dots, n\}$. Given a set X and $k \in \mathbb{N}$, let $\binom{X}{k}$ the family of all k-sets in X. For brevity, we write v for a singleton set $\{v\}$ and xy for a set of pairs $\{x, y\}$. We write $a = b \pm c$ if $b - c \le a \le b + c$. If we claim that a result holds whenever we have $0 < a \ll b, c \ll d < 1$, it means that there exist positive functions f, g such that the result holds as long as $a < f(b, c)$ and $b < g(d)$ and $c < g(d)$. We will not compute these functions explicitly. In many cases, we treat large numbers as if they are integers, by omitting floors and ceilings if it does not affect the argument. We write log for the base-e logarithm.

2.1 Graph notions

For a given path $P = x_1 \dots x_t$, we write $\mathsf{Int}(P) = \{x_2, \dots, x_{t-1}\}$ to denote the set of its internal vertices. For $1 \le i < j \le t$, we write Px_i, x_iPx_j and x_jP resp. for the segments $x_1 \dots x_i$, $x_i \dots x_j$, and $x_j \dots x_t$ resp..

Given a graph G, denote its order and size resp. by $|G|$ and $e(G)$ resp., and its average degree $2e(G)/|G|$ by $d(G)$. For two sets $X, Y \subseteq V(G)$, the (graph) *distance* between them is the length of a shortest path from X to Y. For two graphs G, H, we write $G \cup H$ to denote the graph with vertex set $V(G) \cup V(H)$ and the edge set $E(G) \cup E(H)$. Given a collection of graphs $\mathcal{F} = \{F_i : i \in I\}$, we write $V(\mathcal{F}) = \bigcup_{i \in I} V(F_i)$ and $|\mathcal{F}| = |I|$. For path P and a vertex set U, we write $P|_U$ for the induced subgraph of P on vertex set $V(P) \cap U$.

For $H \subseteq G$, if $V(H) = V(G)$, then H is a *spanning* subgraph of G. Given $S \subseteq V(G)$ and a subgraph $F \subseteq G$, denote by $G - S = H[V(G) \setminus S]$ the subgraph induced on $V(G) \setminus S$, and by $G \setminus F$ the spanning subgraph obtained from G by removing edges in F. Let $e_G(S) = e(G[S])$.

For a set of vertices $X \subseteq V(G)$ and $i \in \mathbb{N}$, denote

$$N^i(X) := \{u \in V(G) : \text{ the distance between } X \text{ and } u \text{ is exactly } i\}$$

the i-th sphere/layer around X, and write $N^0(X) = X$, $N(X) = N^1(X) = \cup_{x \in X} N(x) \setminus X$ for the *external neighbourhood* of X. For $i \in \mathbb{N} \cup \{0\}$, let $B^i(X) = \bigcup_{j=0}^{i} N^j(X)$ be the ball of radius i around X. We write $\partial(X)$ for the edge-boundary of X, that is, the set of edges between X and $V(G) \setminus X$ in G. Given another set $Z \subseteq V(G)$, we write $N(X, Z) = N(X) \cap Z$ for the set of neighbours of X in Z. A less common notation we will also use is $\Gamma(X) = \cup_{x \in X} N(x)$.

For shorthand, we write max-degree/min-degree resp. for the maximum/minimum degree resp..

2.2 Erdős-Rényi binomial random graph

Erdős-Rényi pioneered [6] the study of random graphs.

Definition 2.1 ($G(n, p)$)**.** The *Erdős-Rényi binomial random graph* is a probability space such that when $G \sim G(n, p)$, $V(G) = [n]$ and each pair of vertices is an edge with probability p independent of others.

We write both $G \sim G(n, p)$ and $G = G(n, p)$ when we mean G is drawn from $G(n, p)$.

We say an event happens *almost sure* (a.s.) or *with high probability* (w.h.p.) in $G(n, p)$ if the probability of it occurs tends to 1 as $n \to \infty$.

Definition 2.2 (threshold)**.** The *threshold* of a graph property $\mathcal{P}$ is a function $p_0 = p_0(n)$ such that

$$p = \begin{cases} o(p_0), & G(n, p) \text{ a.s. satisfies } \mathcal{P}; \\ \omega(p_0), & G(n, p) \text{ a.s. does not satisfy } \mathcal{P}. \end{cases}$$

For an example, we leave as exercise to prove the following using first and second moment method (Markov's inequality and Chebyshev's inequality).

Proposition 2.3. *The property of having minimum degree 2 has a (very sharp) threshold:*

$$p = \begin{cases} \frac{\log n + \log\log n - \omega(1)}{n}, & G(n,p) \text{ a.s. has minimum degree at most 1;} \\ \frac{\log n + \log\log n + \omega(1)}{n}, & G(n,p) \text{ a.s. has minimum degree at least 2.} \end{cases}$$

When working with random graphs, the following simple proposition allows us to reveal edges in multi-rounds. We leave the proof as an exercise. Consequently, to prove $G \sim G(n,p)$ a.s. satisfies certain property $\mathcal{P}$, it suffices to show $\cup_{i\in[m]} G_i$ does, where $G_i \sim G(n,p_i)$ with $\sum_{i\in[m]} p_i = p$.

Proposition 2.4. *Given independent $G_1 \sim G(n,p_1)$ and $G_2 \sim G(n,p_2)$, then $G_1 \cup G_2 \sim G(n, p_1 + p_2 - p_1p_2)$.*

We end this section with some useful concentration inequalities. When we have sum of independent random variables, we can use Chernoff.

Lemma 2.5 (Chernoff's inequality)**.** *Let X be sum of independent random variables with expectation μ and $0 \le \delta \le 1$. Then*

$$\Pr(|X - \mu| \ge \delta\mu) \le e^{-\frac{\delta^2\mu}{3}}.$$

Definition 2.6. A *martingale* (*submartingale* resp.) is a sequence of random variables $X_0, X_1, X_2, \ldots$ such that for all $i \ge 0$, $\mathrm{E}|X_i| < \infty$ and $\mathrm{E}(X_{i+1}|X_0,\ldots,X_i) = X_i$ ($\ge X_i$ resp.).

Example 2.7. A natural example of a martingale is a gambler's fortune. Imagine a gambler betting on the outcomes of coin flips, say (s)he wins/loses ¥1 if head/tail. Let X_i be his/her fortune after the ith bet. Then for a fair coin flip, we have $\mathrm{E}(X_{i+1}|X_0,\ldots,X_i) = X_i$.

For martingale with bounded difference, we can use the following.

Lemma 2.8 (Azuma-Hoeffding's inequality)**.** *Let $0 = X_0, X_1, \ldots, X_m$ be a submartingale with $|X_{i+1} - X_i| \le c_i$ for each $0 \le i < m$. Then for all $\lambda > 0$,*

$$\Pr(X_m < -\lambda) \le e^{-\frac{\lambda^2}{2\sum_{i=0}^{m-1} c_i^2}}.$$

3 Depth First Search

Depth First Search (DFS) is a graph exploration algorithm that visits all the vertices of an input graph.

Depth First Search algorithm

Input: a graph G.
Initialisation: pick an arbitrary vertex v in G and define the following three sets.

- $S = (v_1, \ldots, v_s)$: the searching Stack, starting with $s = 1$ and $v_1 = v$;
- $U = V(G) \setminus \{v\}$: the set of Unexplored vertices;
- $X = \varnothing$: the set of eXplored vertices.

Stage 1. Exploration.

- If $N(v_s) \cap U \neq \varnothing$, then add one such neighbour u to the stack:
  $u \to v_{s+1}$, $S \to (v_1, \ldots, v_{s+1})$, $s \to s+1$ and $U \to U \setminus \{u\}$.
  Repeat.
- If $N(v_s) \cap U = \varnothing$, then pop v_s out from the stack to X:
  $X \to X \cup \{v_s\}$, $S \to (v_1, \ldots, v_{s-1})$, $s \to s-1$.
- If $S \neq \varnothing$, start Stage 1 all over. Otherwise go to Stage 2.

Stage 2. Start new component.

- If $U \neq \varnothing$, push $w \in U$ to the stack:
  $w \to v_1$, $S \to (v_1)$, $s \to 1$ and $U \to U \setminus \{w\}$,
  and then go to Stage 1.
  Otherwise terminate the algorithm.

Note that running DFS on a graph G yields a spanning forest, one tree for each component, and there is a natural ordering on the vertices such that the root of each (sub)tree is the first vertex visited within the (sub)tree.

Observation 3.1. The following straightforward observations will be useful.

- The stack S always induces a path in G.
- Each step we either push an unexplored vertex in U to the stack S or pop the last vertex in S to the explored set X. In the former case, $|U|$ decreases by 1, and in the latter case, $|X|$ increases by 1. At the end $|X| = |G|$.
- No edge of G between U and X.
- If an edge uv is not in the forest F from DFS, then one of $\{u, v\}$ is a predecessor of the other in F. In other words,
  - there is no edge of G between subtrees rooted at different vertices; or equivalently
  - say T is a tree in F rooted at r and T' is a subtree rooted at r', then, writting P for the path from r' to r in T, we have $N_G(V(T')) \subseteq V(P)$.

3.1 Long paths in $G(n, p)$, part 1

The following use of DFS to find long paths/cycles is due to Krivelevich, Lee and Sudakov [14].

Theorem 3.2. *$G(n, C/n)$ a.s. has a path of length at least $(1 - O_C(\log C/C))n$.*

The proof consists of two simple steps: first we use DFS to show that m-joined, defined below, graphs has a long path, see Proposition 3.4; then we show that a.s. $G(n, C/n)$ is m-joined for small m, see Proposition 3.5.

Definition 3.3 (Large set connectivity)**.** A graph G is *m-joined* if there is an edge between any two disjoint set of size m. In other words, the complement $\overline{G}$ is $K_{m,m}$-free.

Proposition 3.4. *If an n-vertex G is m-joined, then it has a path of length more than $n - 2m$.*

Proof. Run DFS on G. At some point, $|U| = |X| = x$. As there is no edge between U and X and G is m-joined, we have $x < m$. Now S induces a path of length $n - 2x > n - 2m$. $\square$

Proposition 3.5. *For large C, $G(n, C/n)$ is a.s. m-joined for every $m \geq \frac{3 \log C}{C} n$.*

Proof. It suffices to show that a.s. $G(n, C/n)$ is m-joined for $m = 3n \log C/C$. Taking union bound over all pairs of m-sets not joined, we have for large C that the probability is at most $\binom{n}{m}^2 \cdot (1-p)^{m^2} \leq (en/m)^{2m} e^{-pm^2} \leq (C^2 e^{-3 \log C})^m \to 0$. $\square$

3.2 Long cycles in expander, part 1

An expander is usually refered to a graph whose vertex subsets have expansion property, i.e. having large external neighbourhood. The following is a well-studied one with linear expansion.

Definition 3.6 ((α, k)-expander)**.** Let $k \in \mathbb{N}$ and $\alpha \in \mathbb{R}^+$, a graph G is a (α, k)-*expander* if for every $X \subseteq V(G)$ with $|X| \leq k$,

$$|N(X)| \geq \alpha |X|.$$

Such expander contains a long cycle, linear in αk. For $0 < \alpha < 1$, this is optimal up to the constant factor by considering the complete bipartite graph $K_{\alpha k, k}$ (whose longest cycle is of length $2\alpha k$).

Theorem 3.7. *Let $k \in \mathbb{N}$ and $\alpha \in \mathbb{R}^+$. If G is an (α, k)-expander, then it has a cycle of length at least $\alpha k/2$.*

This theorem is a corollary of the following handy result, due to Alon and Krivelevich [13], which itself is proved by a short but nice analysis on DFS.

Theorem 3.8. *Let $k, t \in \mathbb{N}$ with $t \geq 2$, and G be an n-vertex connected graph with $n > k$ satisfying that for any $W \subseteq V(G)$ with $k/2 \leq |W| \leq k$, $|N(W)| \geq t$. Then G has a cycle of length at least $t + 1$.*

We need the following simple result about trees, whose proof we leave out as exercise.

Proposition 3.9. *Let T be a rooted tree of order at least k. Then there exists a vertex v and a subset X of its children such that the union of the subtrees rooted at X has order between $k/2$ and k.*

Proof of Theorem 3.8. Let T be the tree, say rooted at r, obtained from running DFS on G, and let v and X be as in Proposition 3.9. Let P be the path from v to r in T, and let W be the vertex set of the union of subtrees rooted at X. Then $k/2 \leq |W| \leq k$, and by Observation 3.1, we see that $N_G(W) \subseteq V(P)$. Let $x \in N_G(W) \subseteq V(P)$ be the closest vertex to the root r, and $y \in N_G(x, W)$. Thus, $\{x, y\} \cup V(P)$ contains a cycle of length at least $t+1$ as $|N_G(W)| \geq t$. □

We remark that the result still holds if we drop the connected condition of G by considering its largest component.

4 Pósa's rotation and extension, and edge sprinkling

Pósa [19] introduced a rotation-extension technique for determining the threshold of the appearance of Hamiltonian cycle in binomial random graphs. It is particularly effective to find long paths/cycles when we have linear (at least a factor of 2) expansion. We will cover this technique in this section following the presentation of Montgomery [17].

Pósa's rotation. Let P be a *longest* path in a graph G from say u to v. For each vertex $x \in V(P)$, when exists, write x^- (and x^+ resp.) for the vertex preceding (and following resp.) x on P. For $X \subseteq V(P)$, write $X^- := \{x^- : x \in X\}$, $X^+ := \{x^+ : x \in X\}$ and $X^\pm = X^- \cup X^+$.

- Maximality of P implies that $N(u) \subseteq V(P)$. If $x \in N(u)$, then *break* xx^- and *rotate* P *fixing* v, we get another longest path $P - xx^- + ux$ (on vertex set $V(P)$) from new starting point x^- to v. Call x the *pivot* for this rotation. Again by maximality, $N(x^-) \subseteq V(P)$, we can continue rotating fixing v.

- Let $S(P, v)$ be the set of starting points of the paths that can be obtained from P via a (possibly empty) sequence of rotations fixing v.

Observation 4.1. We will use the following observations.

- All paths derived as above have the same vertex set $V(P)$, so $S = S(P, v) \subseteq V(P)$.
- Maximality of P implies $N(S) \subseteq V(P)$.
- If an edge is broken in a rotation, then one of the endpoints of that edge becomes a starting point. So if an original edge $e \in E(P)$ is broken, then one endpoints of e joins S, while the other joins $S^{\pm}$.

Lemma 4.2 (Pósa's lemma)**.** *Let $P = u \dots v$ be a longest path in a graph G, and $S = S(P, v)$. Then $N(S) \subseteq S^{\pm}$. In particular, $|N(S)| < 2|S|$.*

Proof. Let $x \in S$ and $y \in N(x)$. Then there is a x, v-path P' on $V(P)$. Suffices to show $y \in S \cup S^{\pm}$. Suppose not, then from Observation 4.1, we see that edge(s) incident to y on P was never broken, i.e. $N_{P'}(y) = \{y^-, y^+\}$. But then we can rotate P' pivoting y. Observation 4.1 then implies that $y \in S \cup S^{\pm}$, a contradiction. Since $u \in S$ and it has only one neighbour on P, we have $|N(S)| \leq |S^{\pm}| < 2|S|$. □

Remark 4.3. Pósa's lemma explains where we need 2-expansion. It is usually used as follows. It implies that S does not expand well in G, so if G has expansion for all small sets (e.g. $(2, k)$-expander as in Definition 3.6), then S has to be large ($|S| > k$), i.e. we can get many starting points via rotation. We shall see shortly that how S being large together with large set expansion (Exercise 4.5) implies a long path (Lemma 4.6).

4.1 Long paths in $G(n, p)$, part 2

In this subsection, we sketch how to use the rotation to improve Theorem 3.2.

Theorem 4.4. *$G(n, C/n)$ a.s. has a path of length at least $(1 - O_C(1/C))n$.*

We can find such a long path using large set expansion.

Exercise 4.5 (Large set expansion)**.** Use Chernoff to show that $G(n, C/n)$ a.s. has the property that every set $X \subseteq V(G)$ of size $|X| = m = 10n/C$ has $|N(X)| \geq 4n/5$ for large C.

We remark that this large set expansion is essentially equivalent to the large set connectivity that we have seen in Definition 3.3.

Theorem 4.4 then follows from the following lemma.

Lemma 4.6. *Let G be an n-vertex graph such that for each $X \subseteq V(G)$ of size $|X| = m \leq n/15$, $|N(X)| \geq 4n/5$. Then G contains a path of length at least $n - 2m$.*

Proof sketch. First, by removing a small poorly expanding set, we can bootstrap the expansion of sets of fixed size (m here) to all small sets expansion. More precisely, we leave as an exercise to show the following.

Claim 4.7. There is $W \subseteq V(G)$ with $|W| < m$ such that $G - W$ is a $(2, n/5)$-expander.

Applying Pósa's lemma, Lemma 4.2, to a longest path P in $G - W$, we see that $|S| > n/5$. Maximality of P in $G - W$ implies that there is no edge between S and $Z = V(G - W) \setminus V(P)$, i.e. $N_G(Z) \cap S = \varnothing$ and so $|N_G(Z)| < 4n/5$. The expansion property of G then implies that $|Z| < m$ and so $|P| = n - |W| - |Z| > n - 2m$. □

4.2 Long cycles in expander, part 2

We now give an improvement on Theorem 3.7 when $\alpha \geq 2$.

Theorem 4.8. *Let $k \in \mathbb{N}$ and $\alpha \geq 2$. If G is an (α, k)-expander, then it has a cycle of length at least $(\alpha+1)k$.*

This follows immediately from Lemma 4.2 and the following lemma by Brandt, Broersma, Diestel and Kriesell [5].

Lemma 4.9. *Let $P = u \dots v$ be a longest path in a graph G, and $S = S(P, v)$. Then G has a cycle containing $S \cup N(S)$.*

Proof. Let y be the last vertex on P in $N(S)$. Then in fact y is the last vertex on P in $S \cup N(S)$, i.e. $S \cup N(S)$ is contained in the initial segment Py. Indeed, if there is some $z \in S$ (note $v \notin S$ and so $z \neq v$) lying to the right of y on P, then $z^+ \in N(S)$ would have been the last vertex on P in $N(S)$, contradicting the choice of y. Then, letting $x \in N(y, S)$ be a neighbour of y in S and Q be the x, v-path on $V(P)$, we have by Observation 4.1 that the segment yP was never broken during all rotations and hence $yP = yQ$. This implies that Qy and xy form a cycle containing $S \cup N(S)$. □

4.3 Extension and edge sprinkling

The main result to cover in this subsection is Lemma 4.13, which is a useful tool for finding long cycles in (pseudo)random graphs. Its proof, in a nutshell, is that *rotations* of a maximal path yields many possible *extensions* and then we can *sprinkle* edges in rounds to elongate it.

Extension. Using Pósa's rotation on a longest path P fixing one end in a $(2, n/5)$-expander G, we get at least $n/5$ possible starting points from Lemma 4.2. Fixing these starting points and rotate again, we get $\Omega(n^2)$ longest paths with distinct pairs of endpoints. Closing any one of these paths to a cycle (with vertex set $V(P)$) then allows us to extend P to a longer path as G is connected. This motivates the following notion.

Definition 4.10 (booster)**.** Let $\ell(G)$ be the length of the longest path in G. A pair $uv \in \binom{V(G)}{2}$ is a *booster for G* if $\ell(G + uv) > \ell(G)$ or $G + uv$ is Hamiltonian.

Observation 4.11. To prove that a graph is Hamiltonian, it suffices to show that it can be obtained from adding $|G|$ boosters to some graph G.

As discussed above, we have

Lemma 4.12. *A $(2, n/5)$-expander has at least $\binom{n/5}{2} > n^2/50$ boosters.*

Edge sprinkling. In $G(n,p)$ with $p = \Omega(1/n)$, rather than revealing all edges at once, we can instead sprinkle edges in $\Theta(n)$ rounds, with $\Theta(1)$ edges in each round. If we have $\Omega(n^2)$ boosters as in Lemma 4.12 to begin with, then each round of sprinkling has a positive probability to add a booster. Thus, as pointed out in Observation 4.11, after $\Theta(n)$ rounds, w.h.p. we get a Hamiltonian graph. The following lemma makes this idea precise, saying that sprinkling $\Theta(n)$ edges makes a $(2, n/5)$-expander Hamiltonian.

Lemma 4.13. *Let $G = G(n, 500/n)$ and H_0 be a $(2, n/5)$-expander on $V(G)$. Then $G \cup H_0$ is a.s. Hamiltonian.*

Proof. Let $m = 5n, p = 100/n^2$, and for $i \in [m]$ let $G_i \sim G(n,p)$ be independent and set $H_i = H_0 \cup (\cup_{j=1}^{i} G_j)$. By Proposition 2.4, it suffices to show H_m is a.s. Hamiltonian. Thus, G_i, $i \in [m]$, are the sprinklings on top of H_0 and by Observation 4.11, it remains to show w.h.p. at least n boosters are added during the m rounds of sprinklings.

For this purpose, we define for $i \in [m]$ random variables

$$Y_i = \mathbb{1}_{\{G_i \text{ contains a booster for } H_{i-1}\}}.$$

We want to show $\sum_{i\in[m]} Y_i \geq n$. Note that for each $i \in [m]$, $H_{i-1} \supseteq H_0$ is a $(2, n/5)$-expander and so by Lemma 4.2 contains at least $n^2/50$ boosters. Thus,

$$\mathrm{E}Y_i = \Pr(Y_i = 1) = 1 - \Pr(Y_i = 0) = 1 - (1-p)^{n^2/50} \geq 1 - e^{-pn^2/50} \geq 1/2.$$

Hence, letting $X_0 = 0$ and $X_i = \sum_{j\in[i]}(Y_j - 1/2)$ for $i \in [m]$, we see that $X_0, X_1, \ldots, X_m$ is a submartingale with $|X_{i+1} - X_i| \leq 1$ for each $0 \leq i < m$. By Azuma-Hoeffding, Lemma 2.8, we get $\Pr(X_m < -n) \leq e^{-\frac{n^2}{2m}} = o(1)$ and so a.s. $X_m = \sum_{i\in[m]} Y_i - m/2 \geq -n$ or $\sum_{i\in[m]} Y_i \geq n$ as desired. □

4.4 Finding large $(2, n/5)$-expander in pseudorandom graphs

To apply Lemma 4.13, we need to find a large $(2, n/5)$-expander in $G(n, p)$. We have done so in $G(n, C/n)$ by removing $O(n/C)$ vertices in Exercise 4.5 and Claim 4.7. In this subsection, we show a more economic way, see Lemma 4.15. As usual, we work mainly in a deterministic pseudorandom setting. It is then a routine check that the random graph $G(n, p)$ a.s. satisifies these pseudorandomness conditions.

Essentially the expansion follows from the following properties (corresponding to Lemma 4.15 (i)–(iii) resp.):

- few vertices of small degree and they are mostly far apart (distance 5 suffices);
- large set expansion;
- (*Upper uniform*) no dense patch.

Before stating the lemma, let us first see how one can go about finding large expander in a graph G with the above pseudorandom conditions by removing a small vertex set W. Intuitively, vertices of small degree, call it S, are more likely to expand poorly. Say we want to prove some $U \subseteq V(G)$ expands. By large set expansion property, we may assume that U is a small set. Let $U_1 = U \cap S$ and $U_2 = U \setminus S$. Then

$$|N(U_1 \cup U_2)| \geq (|N(U_1)| - |U_2|) + |N(U_2) \setminus S| - |N(U_1) \cap N(U_2)|.$$

As U_2 consists of vertices of large degree, $U_2 \cup N(U_2)$ has many edges and so it, hence also $N(U_2)$, must be large, for otherwise $U_2 \cup N(U_2)$ is a dense patch. Thus, we need to make sure U_1 expands and the overlap $N(U_1) \cap N(U_1)$ is small.

First of all, for $U_1 \subseteq S$, to have 2-expansion for singletons, we need to remove vertices of degree at most 1. Then, to ensure $|N(U_1)| \geq 2|U_1|$, we can remove paths of length at most 2 with endpoints in S. Next, if each vertex in U_2 has at most one neighbour in $N(U_1)$, we can bound $|N(U_1) \cap N(U_2)| \leq |U_2|$. For this, we can remove paths of length at most 4 with endpoints in S or cycles of length 4 with a vertex in S. Let B_0 be the set of removed vertices so far, then by the 1st condition above B_0 is small.

The set B_0 is almost enough, but not enough. As after removing B_0, some vertices in U_2 might have small degree and so we need to further remove a set B_1 of vertices with large degree to B_0. Note that B_1 is small, for otherwise $B_0 \cup B_1$ is a dense patch. We also want $N(U_1)$ disjoint from B_1, and so we further remove $B_2 = N(B_1) \cap S$. Finally $W = B_0 \cup B_1 \cup B_2$ is the set we want. It is convenient to use the following notion for B_0.

Definition 4.14. Let $S \subseteq V(G)$. An *S-ring* is a path of length at most 4 with endpoints in S or a cycle of length at most 4 intersecting S.

Lemma 4.15. *Let $m, D \geq 4$ and G be an n-vertex graph with $S = \{v : d(v) < D\}$. Suppose G has x vertices of degree at most 1 and y S-rings, and furthermore,*

(i) $x + 7y \leq n/5$ and $y \leq m$;

(ii) for any $A \subseteq V(G)$ with $|A| = m$, $|N(A)| \geq 4n/5$;

(iii) for any $A \subseteq V(G)$ with $|A| \leq 10m$, $e_G(A) < D|A|/100$.

Then there is $W \subseteq V(G)$ with $|W| \leq x + 7y$ such that $G - W$ is a $(2, n/5)$-expander.

Proof. Let B_0 the set of vertices of degree at most 1 or in an S-ring, then $|B_0| \leq x + 5y$ and $|B_0 \setminus S| \leq 3y$. Let $B_1 \subseteq V(G) \setminus (B_0 \cup S)$ be the maximal set with $|B_1| \leq y$ and $e_G(B_0 \cup B_1 \setminus S) \geq D|B_1|/2$. Further set $B_2 = N(B_1) \cap S$. We shall show that $W = B_0 \cup B_1 \cup B_2$ is the desired set.

As $|B_0 \cup B_1 \setminus S| \leq 4y \leq 4m$, if $|B_1| = y$, then $e_G((B_0 \cup B_1) \setminus S) \geq Dy/2 \geq D|B_0 \cup B_1 \setminus S|/8$, contradicting (iii). Thus, $|B_1| < y$. Next, note that every vertex out of B_0 has at most one neighbour in S, so $|B_2| \leq |B_1|$ and $|W| \leq x + 7y$.

Let $H = G - W$ and $U \subseteq V(H)$ with $|U| \leq n/5$, we are left to show $|N_H(U)| \geq 2|U|$. Note that

$$|N_H(v) \setminus S| \geq D/4, \qquad \text{for all } v \in V(H) \setminus S. \tag{1}$$

Indeed, otherwise as $v \notin S \cup B_0$, $d_G(v) \geq D$ and $d_G(v, S) \leq 1$, and so

$$d_G(v, B_0 \cup B_1 \setminus S) \geq d_G(v) - d_G(v, S) - |N_H(v) \setminus S| \geq D - 1 - D/4 \geq D/2.$$

But then v could have been added to B_1, contradicting the maximality of B_1.

Suppose $|U| \geq m$. Taking $U' \subseteq U$ with $|U'| = m$, we have by (i), (ii) and $|U| \leq n/5$ that

$$|N_H(U)| \geq |N_H(U')| - |U| \geq |N_G(U')| - |W| - n/5 \geq 4n/5 - n/5 - n/5 \geq 2|U|.$$

So we may assume $|U| < m$. Write $U = U_1 \cup U_2$, where $U_1 = U \cap S$ and $U_2 = U \setminus S$. Then, as $U_1 \subseteq S$, we have

$$|N_H(U)| = |N_H(U_1 \cup U_2)| \geq (|N_H(U_1)| - |U_2|) + |N_H(U_2) \setminus S| - |N_H(U_1) \cap N_H(U_2)|. \tag{2}$$

We claim that $|N_H(U_2) \setminus S| > 9|U_2|$. Indeed, if not, then $|U_2 \cup N_H(U_2) \setminus S| \leq 10|U_2| \leq 10m$, and as $U_2 \cap S = \varnothing$, by (1), $e_H(U_2 \cup N_H(U_2) \setminus S) \geq D|U_2|/8 \geq D|U_2 \cup N_H(U_2) \setminus S|/80$, contradicting (iii). Recall that as there is no S-ring in H, each vertex in U_2 has at most one neighbour in $N_H(U_1)$ and so $|N_H(U_1) \cap N_H(U_2)| \leq |U_2|$. By (2), it suffices then to show $|N_H(U_1)| \geq 2|U_1|$. This follows from $N_H(u)$, $u \in U_1$, are pairwise disjoint and $|N_H(u) \setminus S| = |N_G(u)| \geq 2$. Indeed, as S-rings were removed, vertices in U_1 have no common neighbour, nor any neighbour in B_0 or $S \supseteq B_2$; and they have no neighbour in B_1 either due to the choice of B_2. □

4.5 Long cycles and Hamiltonicity in $G(n, p)$, part 1

In this subsection, we give two applications of Pósa's rotation, extension and edge sprinkling. The first one, improving further on Theorem 4.4, gives an almost optimal long cycle in $G(n, C/n)$.

Theorem 4.16. *$G(n, C/n)$ a.s. has a cycle of length at least $(1 - e^{-(1-o(1))C})n$.*

The second one determines the sharp threshold of Hamiltonicity in $G(n, p)$, proven first by Bollobás [3] and independently Komlós and Szemerédi [11] in the 80s. As we have seen in Proposition 2.3, when $p = \frac{\log n + \log\log n - \omega(1)}{n}$, $G(n, p)$ a.s. there is a vertex of degree at most 1 and so not Hamiltonian.

Theorem 4.17. *$G(n, p)$ with $p = \frac{\log n + \log\log n + \omega(1)}{n}$ is a.s. Hamiltonian.*

For both theorems above, we split $G(n,p)$ into $G(n, 500/n)$ and $G(n, p-500/n)$ using Proposition 2.4. Then by Lemma 4.13, it suffices to show that $G(n,p)$ a.s. has a large (or spanning) $(2, n/5)$-expander as follows.

Lemma 4.18. *Let C be large and $p = \frac{\log n + \log\log n + \omega(1)}{n}$. Then a.s.*

- *$G(n, C/n)$ has a $(2, n/5)$-expander of order $(1 - e^{-(1-o(1))C})n$;*
- *$G(n,p)$ is a $(2, n/5)$-expander.*

By Lemma 4.15, it then amounts to show that $G(n,p)$ a.s. satisfies (i)–(iii) there with small x, y and m. We leave these routine checks as exercise. Lemma 4.18 follows from the following propositions and Proposition 2.3.

Proposition 4.19. *Let $C \le 2\log n$ be large, $p = C/n$ and $m = n/10^{15}$. Then $G(n,p)$ a.s. satisfies the following.*

- *For any $A \subseteq V(G)$ with $|A| = m$, $|N(A)| \ge 4n/5$.*
- *For any $A \subseteq V(G)$ with $|A| \le 10m$, $e_G(A) < C|A|/10^5$.*

Proposition 4.20. *Let C be large, then $G(n, C/n)$ a.s. satisfies the following.*

- *There are at most $(1 + o_C(1))Ce^{-C}n$ vertices of degree at most 1.*
- *There are at most $e^{-3C/2}n$ S-rings, where $S = \{v : d(v) < C/100\}$.*

Proposition 4.21. *Let $\log n/n \le p \le 2\log n/n$, then $G(n,p)$ a.s. has no S-rings, where $S = \{v : d(v) < np/100\}$.*

Remark 4.22. It is a good time to point out, as seen by Proposition 4.21, that when $p \ge \log n/n$, a.s. S-rings disappear. So the only obstacle to Hamiltonicity is $\delta(G) \le 1$. Bollobás [4] proved the *hitting time* result that considering the random process of edges being uniformly added one by one, then a.s. the graph is Hamiltonian the moment that $\delta(G) \ge 2$.

4.6 Long cycles and Hamiltonicity in $G(n,p)$, part 2

We end this section by mentioning the best possible form of Theorems 4.16 and 4.17.

Theorem 4.23 (Frieze [8])**.** *$G(n, C/n)$ a.s. has a cycle of length at least $(1-(1+o_C(1))Ce^{-C})n$.*

Theorem 4.24 (Ajtai, Komlós, Szemerédi [1])**.** *Let $p = \frac{\log n + \log\log n + c(n)}{n}$ and $c \in \mathbb{R}$, then*

$$\Pr(G(n,p) \text{ is Hamiltonian}) \to \begin{cases} 0, & \text{if } c(n) \to -\infty; \\ e^{-e^{-c}}, & \text{if } c(n) \to c; \\ 1, & \text{if } c(n) \to \infty. \end{cases}$$

We sketch a proof using a conditional argument of Lee and Sudakov [15]. The idea is that if a $(2, n/5)$-expander $H \subseteq G(n,p)$ is sparse, then a.s. G has a booster for H, implying that $G[V(H)]$ is Hamiltonian (see Lemma 4.25). Thus, to find a long cycle, we just need to find a large *and sparse* $(2, n/5)$-expander in $G(n,p)$, which can be done by sparsening large degree vertices (see Lemma 4.26).

Lemma 4.25. *Let $p \ge 10^5/n$. For any $(2, n/5)$-expander $H \subseteq G(n,p)$,*

- *if $e(H) \le pn^2/10^4$, then a.s. $G(n,p)$ has a booster for H;*
- *if $e(H) \le pn^2/10^5$, then $G[V(H)]$ is Hamiltonian.*

Proof. Let $\delta = 10^{-4}$ and $\mathcal{H}$ be the set of all $(2, n/5)$-expanders H on $[n]$ with $e(H) \le \delta pn^2$. By union bound, it suffices to show that $\sum_{H\in\mathcal{H}} \Pr((H \subseteq G) \ \wedge (G \text{ has no booster for } H)) = o(1)$. To bound this sum, for each $H \in \mathcal{H}$, let B_H be the set of boosters in H not in $E(H)$ (for independence later). By Lemma 4.12, $|B_H| \ge n^2/50 - e(H) \ge n^2/100$. Let I_H be the event that $H \subseteq G$, and N_H be the event that $B_H \cap E(G) = \varnothing$, then it suffices to show

$$\sum_{H\in\mathcal{H}} \Pr(I_H \wedge N_H) = \sum_{H\in\mathcal{H}} \Pr(N_H|I_H) \cdot \Pr(I_H) = o(1).$$

In fact I_H and N_H are independent, as $B_H \cap E(H) = \varnothing$. So

$$\Pr(N_H|I_H) = \Pr(N_H) = (1-p)^{|B_H|} \le e^{-pn^2/100}.$$

Therefore,

$$\begin{aligned}\sum_{H\in\mathcal{H}} \Pr(I_H \wedge N_H) &\le e^{-pn^2/100} \sum_{H\in\mathcal{H}} \Pr(I_H) \le e^{-pn^2/100} \sum_{i=0}^{\delta pn^2} \binom{n^2}{i} p^i \le e^{-pn^2/100} \sum_{i=0}^{\delta pn^2} \left(\frac{en^2p}{i}\right)^i \\ &\le e^{-pn^2/100} \cdot n^2 \left(\frac{en^2p}{\delta pn^2}\right)^{\delta pn^2} \le n^2 \cdot e^{-pn^2(\frac{1}{100} - \delta \log \frac{e}{\delta})} \le n^2 \cdot e^{-pn^2/200} = o(1).\end{aligned}$$

□

The following lemma, together with Proposition 4.20, implies Theorems 4.23. Together with Proposition 4.21 and $\Pr(\delta(G(n,p)) \ge 2)$ (exercise), we get Theorem 4.24.

Lemma 4.26 (cf. Lemma 4.15)**.** *Let C be large, $C/n \le p \le 2\log n/n$, and $G = G(n,p)$. Set $S = \{v : d(v) \le pn/100\}$ and let X be the number of vertices of degree at most 1 and Y be the number of S-rings in G. Then a.s. G has a $(2, n/5)$-expander H with order at least $n - X - 7Y$ and $e(H) \le pn^2/10^5$. Consequently, a.s. G has a cycle of length at least $n - X - 7Y$.*

Proof sketch. Basically do the intuitive thing of sparsening G. First, take a random slice $G_1 = G(n, p/10^6)$ for large set expansion and upper uniformity. We do not want to have more small degree vertices, so take G_2 by letting, for each $v \in V(G)$, $d_{G_2}(v) = \min\{d_G(v), np/10^6\}$. Let $G_0 = G_1 \cup G_2 \subseteq G$ and $S_0 = \{v : d_{G_0}(v) \le pn/10^8\}$. Then the number of vertices of degree at most 1 in G_0 is still X, the number of S_0-rings in G_0 is at most Y, and $e(G_0) \le pn^2/10^5$. The conclusion then follows from applying Proposition 4.19 and Lemma 4.15 on G_0 and S_0, and Lemma 4.25. □

5 Friedman-Pippenger theorem on tree embedding

An important result in tree embeddings is by Friedman and Pippenger [7], which finds large trees in graphs with expansion property.

Recall that $\Gamma(X) = \cup_{x\in X} N(x)$.

Theorem 5.1 (Friedman-Pippenger)**.** *Let $d, m \in \mathbb{N}$ and G be a graph. If $|\Gamma(X)| \ge (d+1)|X|$ for all $X \subseteq V(G)$ with $|X| \le 2m$, then G contains every tree with m vertices and max-degree at most d.*

Using the notation of expander in Definition 3.6 and that $N(X) \subseteq \Gamma(X)$, Friedman-Pippenger theorem can be read as follows.

Corollary 5.2. *Let $d, m \in \mathbb{N}$ and G be a $(d+1, 2m)$-expander. Then G contains every tree with m vertices and max-degree at most d.*

5.1 Extendability method

We shall present a proof of Theorem 5.1 using an adaption of Friedman-Pippenger's method due to Glebov, Johannsen and Krivelevich [9]. The idea is to define a *nice* embedding and show that a nice embedding can be extended by adding a leaf (Lemma 5.4).

Definition 5.3 ((d,m)-extendable)**.** Let $d, m \in \mathbb{N}$ with $d \geq 3$ and $m \geq 1$, and G be a graph. A subgraph $S \subseteq G$ is (d,m)*-extendable* if S has max-degree at most d and

$$B(X,S) = |\Gamma_G(X) \setminus V(S)| - \left(d|X| - \sum_{x \in X \cap V(S)} d_S(x) \right) \geq 0,$$

for all sets $X \subseteq V(G)$ with $|X| \leq 2m$. Here, we call $B(X,S)$ the S*-balance* of X, and X is S*-critical* if $|X| \leq 2m$ and $B(X,S) = 0$.

Lemma 5.4. *Let $d, m \in \mathbb{N}$ with $d \geq 3$ and $m \geq 1$, G be a graph and $S \subseteq G$ be a (d,m)-extendable subgraph. Suppose G satisfies*

$$|\Gamma_G(X)| \geq d|X| + |S| + 1,$$

for all $X \subseteq V(G)$ with $m \leq |X| \leq 2m$. Then, for every vertex $s \in V(S)$ with $d_S(s) \leq d-1$, there exists a vertex $y \in N_G(s) \setminus V(S)$ such that the graph $S + sy$ is (d,m)-extendable.

Proof. Let $Y = N_G(s) \setminus V(S)$ and $S_y = S + sy$ for $y \in Y$. As S is (d,m)-extendable and $d_S(s) \leq d-1$, we have $0 \leq B(\{s\}, S) = |Y| - (d - d_S(s)) \leq |Y| - 1$. Thus, Y is non-empty. Suppose for contradiction that none of the S_y, $y \in Y$, is (d,m)-extendable. Then for each $y \in Y$, there is a set $X_y \subseteq V(G)$ with $|X_y| \leq 2m$ such that $B(X_y, S_y) < 0$. Note that

$$B(X_y, S_y) = B(X_y, S) - \mathbb{1}_{y \in \Gamma_G(X_y)} + \mathbb{1}_{s \in X_y} + \mathbb{1}_{y \in X_y}.$$

This follows from $|\Gamma_G(X_y) \setminus V(S_y)| = |\Gamma_G(X_y) \setminus V(S)| - \mathbb{1}_{y \in \Gamma_G(X_y)}$, and

$$\sum_{x \in X_y \cap V(S_y)} d_{S_y}(x) = \sum_{x \in X_y \cap V(S)} d_{S_y}(x) + \mathbb{1}_{y \in X_y} = \sum_{x \in X_y \cap V(S)} d_S(x) + \mathbb{1}_{s \in X_y} + \mathbb{1}_{y \in X_y}.$$

Thus, to have $B(X_y, S_y) < 0$, we must have $B(X_y, S) = 0$, i.e.

$$X_y \text{ is } S\text{-critical}; \quad y \in \Gamma_G(X_y); \quad \text{and} \quad s \notin X_y. \tag{3}$$

We need the following properties of critical sets.

Claim 5.5. All S-critical sets have size at most m, and they are closed under taking union and intersection.

Proof. Let U and V be S-critical, then by definition $|U| \leq 2m$ and

$$|\Gamma_G(U) \setminus V(S)| = d|U| - \sum_{x \in U \cap V(S)} d_S(x) \leq d|U|.$$

Then $|\Gamma_G(U)| \leq |\Gamma_G(U) \setminus V(S)| + |S| \leq d|U| + |S|$, and so the expansion property of G implies that $|U| \leq m$.

For the second part, let $L(U,S) = d|U| - \sum_{x \in U \cap V(S)} d_S(x) = B(U,S) - |\Gamma_G(U) \setminus V(S)|$. Then $L(U \cap V, S) + L(U \cup V, S) = L(U,S) + L(V,S)$. It is not hard to see that $|\Gamma_G(U \cap V) \setminus V(S)| + |\Gamma_G(U \cup V) \setminus V(S)| \leq |\Gamma_G(U) \setminus V(S)| + |\Gamma_G(V) \setminus V(S)|$, and so

$$B(U \cap V, S) + B(U \cup V, S) \leq B(U,S) + B(V,S) = 0. \tag{4}$$

On the other hand, by the first part, $|U|, |V| \leq m$, implying that $|U \cap V|, |U \cup V| \leq 2m$. Then as S is (d,m)-extendable, $B(U \cap V, S), B(U \cup V, S) \geq 0$. So equality holds in (4), and $B(U \cap V, S) = B(U \cup V, S) = 0$ as desired. □

Let $X^* = \cup_{y\in Y} X_y$. Then by Claim 5.5 and (3), X^* is S-critical, and $N_G(s) \setminus V(S) = Y \subseteq \Gamma_G(X^*)$, and $s \notin X^*$. Consequently, $|\Gamma_G(X^* \cup \{s\}) \setminus V(S)| = |\Gamma_G(X^*) \setminus V(S)|$ and $d|X^* \cup \{s\}| = d|X^*| + d$. Therefore, together with X^* is S-critical and $d_S(s) \leq d-1$, we get

$$B(X^* \cup \{s\}, S) = B(X^*, S) - d + \sum_{x\in(X^*\cup\{s\})\cap V(S)} d_S(x) - \sum_{x\in X^*\cap V(S)} d_S(x) = -d + d_S(s) \leq -1.$$

This, however, contradicts S being (d,m)-extendable, as by Claim 5.5, $|X^* \cup \{s\}| \leq m+1 \leq 2m$. □

Theorem 5.1 now follows easily from Lemma 5.4.

Proof of Theorem 5.1. We just need to show that the graph S consisting of a single vertex, i.e. $V(S) = \{s\}$ and $E(S) = \varnothing$, is (d,m)-extendable. Then until S is extended to the full tree, we have $|S| \leq m-1$, and so $|\Gamma_G(X)| \geq (d+1)|X| \geq d|X| + |S| + 1$, for all $X \subseteq V(G)$ with $m \leq |X| \leq 2m$. Thus, we can embed the full tree by adding one leaf at a time using Lemma 5.4.

So let $V(S) = \{s\}$ and take $X \subseteq V(G)$ with $|X| \leq 2m$. Then, as $E(S) = \varnothing$, $d_S(x) = 0$ for all $x \in X$, and so $B(X,S) = |\Gamma_G(X) \setminus \{s\}| - d|X| \geq (d+1)|X| - 1 - d|X| \geq 0$. □

5.2 Large trees in expanders with large set connectivity

Note that Theorem 5.1 can only find trees of size up to $|G|/2(d+1)$. How about larger trees? Re-examine the proof of Theorem 5.1, we see that the same proof yields larger trees as follows.

Theorem 5.6. *Let $d, m \in \mathbb{N}$. Suppose that G is a graph satisfying:*

- $|\Gamma_G(X)| \geq d|X| + 1$, *for every $X \subseteq V(G)$ with $1 \leq |X| \leq m$; and*
- $|\Gamma_G(X)| \geq d|X| + t$, *for every $X \subseteq V(G)$ with $m+1 \leq |X| \leq 2m$.*

Then G contains a copy of every tree T with $|T| \leq t$ and max-degree at most d as a subgraph. Furthermore, if T is a rooted tree, then, for any $v \in V(G)$, we can embed T in G with v as its root.

Remark 5.7. The 'furthermore' part above could be useful when we want to attach a copy of some tree to a particular vertex say v to enlarge its 'boundary'. How can this be useful? If our task is to link v to say some set A, what we usually do is to expand v to try to reach A. If a large tree T is already attached to v, then expanding $V(T)$ is a much easy task than expanding v from scratch. Moreover, we can choose the shape of our tree by e.g. controlling its depth so that we can have a say about the length of the v, A-path.

If we want to embed large T, the 2nd condition above for large t is basically equivalent to large set expansion (as in Exercise 4.5) or large set connectivity (as in Definition 3.3).

In terms of expanders (as in Definition 3.6), Theorem 5.6 can be rephrased as follows. It finds almost spanning bounded degree trees in expanders with large set connectivity. For $t, \Delta > 0$, let $\mathcal{T}(t,\Delta)$ be the family of all trees with t vertices and max-degree at most Δ. A graph is $\mathcal{T}(t,\Delta)$-*universal* if it contains a copy of every tree in $\mathcal{T}(t,\Delta)$ *simultaneously.*

Theorem 5.8. *Let $n, \Delta \in \mathbb{N}$, $d \in \mathbb{R}^+$ with $d \geq 2\Delta$, and G be an n-vertex $\frac{n}{2d}$-joined $(d, \frac{n}{2d})$-expander. Then G is $\mathcal{T}(n - 4\Delta\frac{n}{2d}, \Delta)$-universal.*

We leave the derivation of Theorem 5.8 from Theorem 5.6 as an exercise.

5.3 Almost spanning bounded degree trees in $G(n,p)$

Let us now consider embedding almost spanning bounded degree tree in $G(n,p)$. It is known that when $p < \frac{1-\varepsilon}{n}$, then a.s. all connected components of $G(n,p)$ has logarithmic size. As we have seen, when $p \geq C/n$ for large C, we a.s. get almost spanning cycle; there is no obvious reason why we cannot get a given almost spanning bounded degree tree. Indeed, Alon, Krivelevich and Sudakov [2] showed that the threshold of appearance of *almost* spanning trees is $\Theta(1/n)$. In fact, they showed that for large C, $G(n, C/n)$ is a.s. $\mathcal{T}((1-o(1))n, \Delta)$-*universal*.

Theorem 5.9. *Let $\varepsilon, \Delta > 0$. Then there exists $C = C(\varepsilon, \Delta)$ such that a.s. $G = G(n, C/n)$ is $\mathcal{T}((1-\varepsilon)n, \Delta)$-universal.*

By Theorem 5.6 (with $|T| = (1-o(1))n$), to prove Theorem 5.9, we just need to show that $G(n,p)$ a.s. has large set expansion (as in Exercise 4.5) and bootstrap it to small set expansion (as in Claim 4.7).

Proof of Theorem 5.9. Let $1/C \ll \varepsilon, \Delta$. By Chernoff, a.s. we have, for every $X \subseteq V(G)$ with $|X| = m = \frac{10\log(1/\varepsilon)}{C} \cdot n$, that $|N_G(X)| \geq (1-\varepsilon/2)n$.

Let $W \subseteq V(G)$ be a maximal set such that $|W| \leq 2m$ and $|N_G(W)| < (\Delta+1)|W|$. We claim that $|W| < m$. Indeed, if not, take $W' \subseteq W$ of size $|W'| = m$, then the above expansion property implies that

$$|N_G(W)| \geq |N_G(W')| - |W| \geq (1-\varepsilon/2)n - 2m \geq n/2 > (\Delta+1)2m \geq (\Delta+1)|W|,$$

a contradiction.

Suppose there is a set $U \subseteq V(G-W)$ with $|U| \leq m$ and $|N_{G-W}(U)| < (\Delta+1)|U|$. Then $|N_G(W \cup U)| < (\Delta+1)|W \cup U|$. But $|W| < m$ and so $|W \cup U| \leq 2m$, contradicting the maximality of W.

Thus, $G - W$ satisfies the first condition in Theorem 5.6 (with $d = \Delta$). For the other one, take $X \subseteq V(G)$ with $m+1 \leq |X| \leq 2m$ and let $X' \subseteq X$ be of size m, we have

$$|N_G(X)| \geq |N_G(X')| - |X| \geq (1-\varepsilon/2)n - 2m \geq \Delta \cdot 2m + (1-\varepsilon)n \geq \Delta|X| + |T|.$$

Thus by Theorem 5.6, $G-W$, hence also G, contains a copy of T for every $T \in \mathcal{T}((1-\varepsilon)n, \Delta)$. □

6 Spanning bounded degree trees in $G(n,p)$

Note that the picture changes when we ask for spanning trees instead of just almost spanning one. As a necessary condition for having spanning tree is connectivity, which has threshold $p = \log n/n$. More precisely, $G(n,p)$ needs $p = (\log n + \omega(1))/n$ to be a.s. connected (which coincides with when $\delta(G) \geq 1$). Montgomery [18] determined the threshold of appearance of bounded degree spanning trees, in fact, he showed that $G(n, C\log n/n)$, $C = C(\Delta)$, a.s. is $\mathcal{T}(n, \Delta)$-universal. Here we present a weaker bound for a given tree from [16].

Theorem 6.1. *Let T be an n-vertex tree with max-degree Δ. Then a.s. $G(n,p)$ with $p = \Delta \log^5 n/n$ contains a copy of T.*

We shall distinguish trees into two kinds and treat them separately. We say a path in a tree T is a *bare path* if all its internal vertices have degree 2 in T. Then a tree is *leafy* if it has many leaves and *leggy* if it has many bare paths. The following lemma makes it precise.

Lemma 6.2 (Krivelevich [12])**.** *Let $n, k > 2$ be integers. An n-vertex tree either has at least $\frac{n}{4k}$ leaves or at least $\frac{n}{4k}$ vertex disjoint bare paths, each of length k.*

We briefly outline the ideas here. Let T be an n-vertex bounded degree tree and let $k = \Theta(\log^2 n)$. If T is leafy, remove its leaves to get T', embed T' using the almost spanning tree result, and reveal more edges to match the remaining vertices as leaves to the right vertices in T'. If T is leggy, remove $\frac{n}{4k}$ length-k bare paths to get a forest T', embed T' using the almost spanning tree result, then the remaining task is to link $\frac{n}{4k}$ pairs of vertices with length-k paths. This very last step of linking is done utilising expansions.

It is worth pointing out that [16] differs from previous work [12] at the last linking step. Previously k was taken to be constant $k = \Theta(1)$ and the linking was done via a result of Johansson, Kahn and Vu [10]; while now k is larger, allowing longer paths to be used, which can be done at a lower probability.

We now collect some basic properties of random graphs before giving the proof of Theorem 6.1. First we need a notion which captures both the small set expansion in Definition 3.6 and large set connectivity in Definition 3.3.

Definition 6.3. For a graph G and a set $W \subseteq V(G)$, we say G *d-expands into W* if

- $|N_G(X, W)| \geq d|X|$ for all $X \subseteq V(G)$ with $|X| \leq \frac{|W|}{2d}$, and
- $e_G(X, Y) > 0$ for all disjoint $X, Y \subseteq V(G)$ with $|X| = |Y| = \frac{|W|}{2d}$.

So for example if an n-vertex graph G d-expands into $V(G)$, then G is $\frac{n}{2d}$-joined $(d, \frac{n}{2d})$-expander. As we have seen before, random graphs naturally expand well.

Lemma 6.4. *Let $d : \mathbb{N} \to \mathbb{R}^+$ satisfy $d \geq 3$. Then $G = G(n, \frac{7d \log n}{n})$ a.s. d-expands into $V(G)$.*

The next lemma allows us to partition an expander such that it expands well into each part.

Lemma 6.5. *Let $k \in \mathbb{N}$ and $d \in \mathbb{R}^+$. The following holds for n sufficiently large and $k \leq \log n$. Let $m, m_1, \ldots, m_k \in \mathbb{N}$ satisfy $m = m_1 + \cdots + m_k$ and let $d_i = \frac{m_i}{m} d$ satisfy $d_i \geq 2 \log n$ for all $i \in [k]$. Suppose a graph G d-expands into a set W with $|W| = m$. Then there is a partition $W = W_1 \cup \ldots \cup W_k$ such that, for each $i \in [k]$, $|W_i| = m_i$ and G d_i-expands into W_i.*

The following lemma finds a star matching in random graphs. It will be used to attach the removed leaves for leafy trees.

Lemma 6.6. *Let $d_1, \ldots, d_k \in \mathbb{N}$ with $d_i \leq \Delta$ and $\sum_{i \in [k]} d_i = \ell$. Let $A = \{a_1, \ldots, a_k\}$ and B be disjoint vertex sets with $|B| = \ell$. Let G be a randdom bipartite graph with parts A and B with edge probability $p \geq \frac{2\Delta \log \ell}{\ell}$. Then a.s. as $\ell \to n$, G contains vertex disjoint stars $S_1, \ldots, S_k$ such that, for each $i \in [k]$, S_i is centered at a_i with d_i leaves in B.*

Proof. Blow up G to balanced bipartite graph and find a perfect matching there. That is, for each $i \in [k]$, replace a_i with A_i with $|A_i| = d_i$, and add edges between A_i and B with probability p_i with $(1 - p_i)^{d_i} = 1 - p$, implying that $p_i \geq p/d_i \geq p/\Delta$. Call the resulting graph G'. Note that the distribution of G' induces that of G in the obvious way: $a_i b \in E(G)$ if and only if $e_{G'}(A_i, b) > 0$. So a perfect matching in G' corresponds to the desired star matching in G. Recall that when each edge in G' appears with probability at least $\frac{\log \ell + \omega(1)}{\ell}$, then a.s. there is a perfect matching. We are done as $p_i \geq p/\Delta \geq \frac{2 \log \ell}{\ell}$. $\square$

6.1 Covering expanders with paths

Following the outline at the beginning of the section, the proof of Theorem 6.1 reduces to the following result on covering expanders with paths of specified ends and lengths.

Theorem 6.7. *Let n be sufficiently large and $\ell \in \mathbb{N}$ satisfy $\ell \geq 10^3 \log^2 n$ and $\ell \mid n$. Let G be an n-vertex graph containing $\frac{n}{\ell}$ disjoint vertex pairs (x_i, y_i) and let $W = V(G) \setminus (\cup_i \{x_i, y_i\})$. Suppose G d-expands into W with $d = \frac{10^{10} \log^4 n}{\log \log n}$. Then we can cover $V(G)$ with $\frac{n}{\ell}$ x_i, y_i-paths P_i, each of length $\ell - 1$.*

Proof of Theorem 6.1. We treat leafy and leggy trees separately. Let T be an n-vertex tree with max-degree at most Δ, and let $k = 10^3 \log^2 n$. Then by Lemma 6.2, either T has at least $\frac{n}{4k}$ leaves or at least $\frac{n}{4k}$ vertex disjoint bare paths of length k each.

Leafy T. If T has at least $\frac{n}{4k} = \Theta(n/\log^2 n)$ leaves, then remove $\frac{n}{4k}$ leaves from T to get T'. Let $G_1, G_2 \sim G(n, \frac{\Delta \log^5 n}{2n})$, then a.s. $G_1 \cup G_2 \subseteq G$. By Lemma 6.4, G_1 (hence also G) a.s. $\frac{\Delta \log^4 n}{20}$-expands into $V(G)$, and so by Theorem 5.8, $T' \subseteq G_1$. Then by Lemma 6.6 with $\ell = \frac{n}{4k}$, we can embed the missing leaves to T' to get T using edges from G_2.

Leggy T. If T has at least $\frac{n}{4k}$ bare paths with length k each, then remove the internal vertices of such bare paths to get T'' with $|T''| < 5n/6$. Recall that G $\frac{\Delta \log^4 n}{20}$-expands into $V(G)$, so by Lemma 6.5 with $W = V(G)$, we get $W = W_1 \cup W_2$ such that $|W_1| = 7n/8$, $|W_2| = n/8$ and G $\frac{\log^4 n}{200}$-expands into each W_i, $i \in [2]$. Again by Theorem 5.8, $T'' \subseteq G[W_1]$. To finish embedding T, we just need to cover the remaining vertices, say Z, with x_i, y_i-paths, each of length k, where x_i, y_i are the vertices in the partial embedding corresponding to the endpoints of the bare paths. Since $Z \supseteq W_2$, so G also expands well into Z and we can use Theorem 6.7 to cover Z with the desired missing paths. □

To cover expanders with paths of specified ends and lengths as in Theorem 6.7, we need two ingredients:

(i) a way to link given vertex pairs by paths of specified length, see Lemma 6.8;

(ii) the *absorption method* introduced by Rödl, Ruciński and Szemerédi, see Lemma 6.10.

Roughly speaking, (i) covers most of the vertices and then (ii) allows us to incorporate the leftover vertices into the paths.

To say a few words about the absorption method, it is a general method that is useful for finding spanning substructure, say F, in a host graph G. The basic senario where we can employ this method is when we can find almost spanning substructure (part of F, say F') robustly in G. If so, we first build some *absorber* A which can absorb any small subset of some designated set R. Then we set A aside and find F' in $G - A$ with leftover vertices lying in R. Finally, using A we absorb the leftover in R to cover $V(G)$ and turn F' into F.

6.2 Linking with specified ends and lengths

The first ingredient below finds paths in expanders with given ends and lengths as in Theorem 6.7, except that only up to $3/4$ of W can be covered.

Lemma 6.8. *Let n be sufficiently large and $d = \frac{160 \log^2 n}{\log \log n}$. Let G be an n-vertex graph containing disjoint sets X, Y, W with $X = \{x_i\}_{i \in [r]}$, $Y = \{y_i\}_{i \in [r]}$. For $i \in [r]$, let $k_i \in \mathbb{N}$ with $\frac{4 \log n}{\log \log n} \leq k_i \leq \frac{n}{40}$ and $\sum_{i \in [r]} k_i \leq 3|W|/4$. If G d-expands into W, then there are x_i, y_i-paths P_i, $i \in [r]$, with internal vertices in W and length k_i.*

We shall start with the following easier task, which, given large set expansion, links at least one pair among many pairs.

Lemma 6.9. *Let $m, n \in \mathbb{N}$ with $m \leq \frac{n}{800}$, $d = \frac{n}{200m}$ and n sufficiently large. Let G be an n-vertex graph such that any set $A \subseteq V(G)$ with $|A| = m$ satisfies $|N(A)| \geq (1 - \frac{1}{64})n$. Let X, Y, U be disjoint sets with $X = \{x_i\}_{i \in [2m]}$, $Y = \{y_i\}_{i \in [2m]}$ and $|U| = \frac{n}{8}$. For $i \in [2m]$, let $k_i \in \mathbb{N}$ with $\frac{2 \log n}{\log d} \leq k_i \leq \frac{n}{40}$. Then for some $i \in [2m]$, there is an x_i, y_i-path with internal vertices in U and length k_i.*

Proof sketch. Note first that as $|X| = |Y| = 2m$, by large set expansion, their neighbourhoods in U intersect, so there is a path between some x_i and y_j. What we have to do is (1) matching the index, i.e. $i = j$, so the path is between a given pair; (2) lengthening the path to have the correct length.

We first prepare the graph. Using Lemma 6.5, partition U into U_1, U_2 of equal size with G expanding into both of them. Using large set expansion, we can remove a small set (size at most m) from U_i to get V_i with small set expansion (as in the proof of Theorem 5.9).

Now, for (1), the idea is averaging and pigeonhole. Averaging: by large set expansion, every m-set in X has large neighbourhood in V_1; then by averaging, there is a vertex that expands well into V_1, implying that there are at least $m+1$ vertices in X that expand into V_1. Pigeonhole: the same holds for Y w.r.t. V_2, then there must exists some $i \in [2m]$ such that x_i and y_i expand into V_1 and V_2 resp.. Then both $H_1 = G[V_1 \cup \{x_i\}]$ and $H_2 = G[V_2 \cup \{y_i\}]$ expand well.

For (2), the idea is to attach an appropriate tree T_1 (see Remark 5.7) to x_i in H_1 with depth about $k_i/2$ and leaf set say L_1 of size m. Do the same for y_i in H_2 to get T_2 with leaf set L_2. Then a final application of large set expansion to L_1, L_2 to get the length-k_i x_i, y_i-path in U. □

Proof sketch for Lemma 6.8. Note first that Lemma 6.9 implies that

($*$) for any $4m$ X, Y-pairs, we can match up 1/2 of them with paths of desired lengths.

In particular, we may assume that there are $2m$ unmatched pairs, say $X_1 \subseteq X, Y_1 \subseteq Y$. We will match these $2m$ pairs in $k = \log_2 m + 1$ rounds with each round matching 1/2 of what remains. To do this, using Lemma 6.5, partition W into W_0, W_i, W_i', $i \in [k]$, with $|W_0| = 9|W|/10$ and all W_i, W_i' equal size, such that G expands into each part.

Now take a 2-matching from X_1 to W_1, we get a $4m$-set. Do the same for Y_1 w.r.t. W_1', then by ($*$), we can match 1/2 of them (hence also 1/2 of X_1, Y_1) in W_0. Let X_2, Y_2 be the unmatched set. We then repeat this, i.e. take 2-matching from X_2 to W_2... After $k = \log_2 m + 1$ rounds, everything is matched. □

6.3 Absorber from expansion

The second ingredient below finds absorber (W') that can absorb the leftover (A').

Lemma 6.10. *Let n, r be sufficiently large, $\ell = 10^3 \log^2 n$. Let G be an n-vertex graph containing disjoint sets A, X, Y, W with $X = \{x_i\}_{i \in [3r]}$, $Y = \{y_i\}_{i \in [3r]}$, $|A| = 2r \leq \frac{|W|}{3\ell}$. If G $400 \log^2 n$-expands into W, then there is $W' \subseteq W$ such that for any $A' \subseteq A$ with $|A'| = r$, there are vertex disjoint x_i, y_i-paths, $i \in [3r]$, each of length $\ell - 1$, covering $W' \cup A'$.*

To get absorber for a subset of vertices, we will chain together absorbers for single vertex, defined as follows.

Definition 6.11. An *absorber* (R, x, y) for a vertex v in a graph G is such that both $G[R]$ and $G[R \cup \{v\}]$ have a spanning x, y-path. We call $|R|$ the *size* of the absorber and x, y its *ends*.

Using expansion properties, the following lemma finds absorbers for single vertex.

Lemma 6.12. *Let n be sufficiently large and $d = 20 \log^2 n$. Let G be an n-vertex graph containing disjoint sets A, W with $|A| \leq \frac{|W|}{300 \log^2 n}$. If G d-expands into W, then we can find in $G[W]$ disjointly 40 absorbers for each vertex in A, each of size $\log^2 n + 2$.*

Proof sketch. Take $v \in A$. Let us first find one absorber in W for v. Partition, using Lemma 6.5, W into W_1, W_2, W_3 of equal size. Take a 2-matching from v to W_1, say with $x_0, y_1 \in N(v, W_1)$. Let $k = \log n$, then applying Lemma 6.8 twice: first in W_2 to get a length-$(2k+1)$ x_0, y_1-path $Q = x_0 x_1 x_2 \ldots x_{k-1} x_k y_0 y_k y_{k-1} \ldots y_2 y_1$; then in W_3 to get disjoint x_i, y_i-paths P_i, $i \in [k]$, of length $(k-1)$ each. Then by winding around Q using P_i, it is not hard to see that, letting $R = \cup_{i \in k} V(P_i) \cup \{x_0, y_0\}$, (R, x_0, y_0) is an absorber for v. Taking $k = 3$ for instance, the x_0, y_0-path in R is $x_0 x_1 P_1 y_1 y_2 P_2 x_2 x_3 P_3 y_3 y_0$; and the x_0, y_0-path in $R \cup \{v\}$ is $x_0 v y_1 P_1 x_1 x_2 P_2 y_2 y_3 P_3 x_3 y_0$.

To get 40 disjoint absorbers for all vertices in A, we instead start with taking a 80-matching from A to W_1 and link appropriate pairs in $W_2 \cup W_3$ as above. $\square$

Now the natural thought to get Lemma 6.10 is to take one absorber for each vertex in A and distribute them to X, Y-pairs. That is, find absorbers (R_i, a_i, b_i) in W, $i \in [2r]$, one for each vertex in A, then use Lemma 6.8 to link pairs $(x_i, a_i), (b_i, y_i)$. Then, no matter what $A' \subseteq A$ is given, we have x_i, y_i-paths absorbing it. The problem here is that only some $|A'|$ paths get to absorb a vertex, hence we have no precise control on the lengths of x_i, y_i-paths constructed.

To have more flexibility to fix the above issue, we can chain the absorbers up so that each chain can absorb a subset of A. Imagine now an auxiliary bipartite graph H with one partite set being the chains and the other being A, and the neighbourhood of a chain in H is the subset of A that it can absorb. Then the task of grouping up the absorbers into chains amounts to the following 'resilient matching' statement.

Lemma 6.13. *For sufficiently large n, there is a bipartite graph H on partite set X and $Y \cup Z$ with $|X| = n$, $|Y| = |Z| = 2n/3$, and max-degree 40, such that the following holds. For any $Z' \subseteq Z$ with $|Z'| = n/3$, there is an $X, Y \cup Z'$-matching.*

Proof sketch. Take two disjoint sets X_1, Y, each of size $2n/3$. Let G be a union of 20 independent random X_1, Y-matchings. Clone vertices in Y to get Z and clone $n/3$ vertices in X_1 to get X_2. Let $X = X_1 \cup X_2$. Then it can be shown that w.h.p. the bipartite graph H on X and $Y \cup Z$ is as desired. $\square$

Proof sketch of Lemma 6.10. Partition, using Lemma 6.5, W into W_1, W_2, W_3 of equal size. Take $B \subseteq W_1$ with $|B| = |A|$. Apply Lemma 6.12 to get disjointly 40 absorbers in W_2 for each vertex in A. We group them up into $3r$ chains, say $C = \{c_i\}_{i \in [3r]}$, using H from Lemma 6.13 with $(X, Y, Z)_{6.13} = (C, B, A)$. Each chain c_i is obtained from grouping up one absorber for each $v \in N_H(c_i)$. We claim that $W' = V(C) \cup B$ is as desired.

To see this, note that for any $A' \subseteq A$ with $|A'| = r$, H has a $C, A' \cup B$-matching, which means that each chain gets to absorb exactly one vertex. Then incorporate the $3r$ chains to X, Y-pairs, one for each pair using Lemma 6.8, we get X, Y-paths of prescribed length covering $A' \cup W'$. $\square$

6.4 Proof sketch of Theorem 6.7

By chopping paths into shorter ones (via taking a matching in W and link also the endpoints of the matching), we may assume $\ell = 10^3 \log^2 n$.

Set $m = \frac{n}{2d} \ll s = \frac{n}{10^5 \log^3 n}$ and $r = 2s \log n = \Theta(\frac{n}{\log^2 n})$. Partition, using Lemma 6.5, W into W_1, W_2, W_3 with $|W_1| = r/2, |W_2| = 3r/2$ and $|W_3| = n - o(n)$. Let $W' \subseteq W_3$ from Lemma 6.10 with $(A, W)_{6.10} = (W_1 \cup W_2, W_3)$. Let $I = [3r + 1, n/\ell]$. We will set W' aside and link pairs (x_i, y_i), $i \in I$, using (x_i, y_i)-paths P_i that cover the whole $W \setminus W'$ except some $W'' \subseteq W_1 \cup W_2$ with $|W''| = r$. Then the property of W' implies that there are (x_i, y_i)-paths, $i \in [3r]$, that cover the remaining set $W' \cup W''$, finishing the proof. We are left to find such P_i, $i \in I$.

We shall (again) chop P_i into segments of length $\ell_0 = 2 \log n + 2$ as follows. Take disjointly in W_3 a set $A_i = \{a_{i,j}\}_{j \in [k]}$ of $k = \frac{\ell}{\ell_0} - 1$ vertices for each $i \in I$. Then to get P_i, we just need to link $(x_i, a_{i,1}), (a_{i,k}, y_i)$ and $(a_{i,j}, a_{i,j+1})$ for $j \in [k-1]$ with paths of length ℓ_0 each.

By Lemma 6.9, we can link all but at most $O(m) \ll s$ such pairs. Let us leave s pairs unmatched. Call the endpoints of these s unmatched pairs U. By the choices of the parameters, one can see that there are exactly s vertices in W_3 left uncovered by the paths found so far. Call this set of uncovered vertices U'. We can then take a matching from U to W_1 and a 2-matching from U' to W_1. By Lemma 6.8, we can link the (right) pairs of the endpoints of these matchings in W_1 using W_2. This finishes all the P_i, $i \in I$. A simple calculation shows that exactly a set W'' of r vertices in $W_1 \cup W_2$ left uncovered as desired.

References

[1] M. Ajtai, J. Komlós, E. Szemerédi, First occurence of Hamilton cycles in random graphs. *North-Holland Mathematics Studies*, **115**(C) (1985), 173–178.

[2] N. Alon, M. Krivelevich, B. Sudakov, Embedding nearly-spanning bounded degree trees. *Combinatorica*, **27**(6) (2007), 629–644.

[3] B. Bollobás, The evolution of random graphs. *Transactions of the American Mathematical Society*, **286**(1) (1984), 257–274.

[4] B. Bollobás, The evolution of sparse graphs. *Graph Theory and Combinatorics* (Cambridge 1983), (1984), 35–57.

[5] S. Brandt, H. Broersma, R. Diestel, M. Kriesell, Global connectivity and expansion: long cycles and factors in f-connected graphs. *Combinatorica*, **26** (2006), 17–36.

[6] P. Erdős, A. Rényi, On random graphs I. *Publicationes Mathematicae* (Debrecen), **6** (1959), 290–297.

[7] J. Friedman, N. Pippenger, Expanding graphs contain all small trees. *Combinatorica*, **7**(1) (1987), 71–76.

[8] A. Frieze, On large matchings and cycles in sparse random graphs. *Discrete Mathematics*, **59**(3) (1986), 243–256.

[9] R. Glebov, On Hamilton cycles and other spanning structures. *PhD thesis*, (2013).

[10] A. Johansson, J. Kahn, V. Vu, Factors in random graphs. *Random Structures and Algorithms*, **33**(1) (2008), 1–28.

[11] J. Komlós, E. Szemerédi, Limit distribution for the existence of Hamiltonian cycles in a random graph. *Discrete Mathematics*, **43**(1) (1983), 55–63.

[12] M. Krivelevich, Embedding spanning trees in random graphs. *SIAM Journal on Discrete Mathematics*, **24**(4) (2010), 1495–1500.

[13] M. Krivelevich, Long cycles in locally expanding graphs, with applications. *Combinatorica*, **39**(1) (2019), 135–151.

[14] M. Krivelevich, C. Lee, B. Sudakov, Long paths and cycles in random subgraphs of graphs with large minimum degree. *Random Structure and Algorithms*, **46**(2) (2015), 320–345.

[15] C. Lee, B. Sudakov, Dirac's theorem for random graphs. *Random Structure and Algorithms*, **41**(3) (2012), 293–305.

[16] R. Montgomery, Embedding bounded degree spanning trees in random graphs. *arXiv: 1405.6559*, (2014).

[17] R. Montgomery, Topics in random graphs. *Lecture notes*, (2018).

[18] R. Montgomery, Spanning trees in random graphs. *Advances in Mathematics*, **356**, 106793, (2019).

[19] L. Pósa, Hamiltonian circuits in random graphs. *Discrete Mathematics*, **14**(4) (1976), 359–364.