Shape of the asymptotic maximum sum-free sets in integer lattice grids

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Abstract

We determine the shape of all sum-free sets in $\{1,\dots,n\}^2$ of size close to the maximum $\frac{3}{5}n^2$, solving a problem of Elsholtz and Rackham. We show that all such asymptotic maximum sum-free sets lie completely in the stripe $\frac{4}{5}n - o(n) \le x + y \le \frac{8}{5}n + o(n)$. We also determine for any positive integer p the maximum size of a subset $A \subseteq \{1,\dots,n\}^2$ which forbids the triple (x,y,z) satisfying $px + py = z$.

1 Introduction

A cornerstone result of Schur [13] states that for sufficiently large integer n and a fixed integer r, any r-coloring of $[n] := \{1, ..., n\}$ yields a monochromatic triple x, y, z such that $x + y = z$. For an integer $n \in \mathbb{N}$ a subset $A \subseteq [n]$ is *sum-free* if it has no solution for the equation $x + y = z$, i.e. for all $x, y \in A$ we have $x + y \notin A$. The topic of sum-free sets of integers is well-studied in combinatorial number theory and has a long history.

It is clear that the sets

$$S_1 = \left\{1, 3, 5, \dots, 2\left\lfloor \frac{n-1}{2} \right\rfloor + 1\right\} \quad \text{and} \quad S_2 = \left\{\left\lceil \frac{n+1}{2} \right\rceil, \left\lceil \frac{n+1}{2} \right\rceil + 1, \dots, n\right\}$$

are sum-free and of size $\left\lceil \frac{n}{2} \right\rceil$. If n is even, $S_3 = S_2 - 1$ is another one of the same size. Let us denote the density of a maximum sum-free subset of $[n]$ by $\mu([n]) := \max\{\frac{|S|}{n} \mid S \subseteq [n], S \textit{ is sum-free}\}$. If $S \subseteq [n]$ is a sum-free set and $a \in S$ is the largest element, then at most one of x or $a - x$ can be in S for each $x \le a$. Therefore $|S| \le \left\lceil \frac{a}{2} \right\rceil \le \left\lceil \frac{n}{2} \right\rceil$. Together with the above examples, we see that

$$\mu([n]) = \begin{cases} \frac{1}{2} & \text{if } n \text{ even,} \\ \frac{1}{2} + \frac{1}{2n} & \text{if } n \text{ odd.} \end{cases}$$

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1.1 Structure for large sum free sets

Given the extremal result, great efforts has been made to better understand the general structure of large sum-free sets in $[n]$. The first result on this topic was due to Freiman [8] who showed that if the size of a sum-free set in $[n]$ is large enough, then it will either consist of all odd numbers as in S_1 above or it will be close to second half of the interval as S_2. We remark that more structural results are known for large sum-free sets in 1-dimensional integer lattice (see [6] and a recent progress [15]). Such structual results are not only interesting on their own; they have been utilized e.g. in recent work on enumerating maximal sum-free sets (see [1]).

The problem of sum-free sets has been generalized to higher dimensional lattice $\mathbb{Z}^d$, $d \geq 2$. Similarly, we define $\mu([n]^d) := \max\{\frac{|S|}{n^d} \mid S \subseteq [n]^d \textit{ is sum-free}\}$. In particular, for $d = 2$, the problem of finding the largest sum-free subset of $[n]^2 = \{1, 2, \cdots, n\}^2$ was firstly presented by Cameron as an unsolved problem in [4].

Conjecture 1.1. [4] *There exists a constant $c \in \mathbb{R}$ such that $\mu([n]^2) = c + O(1/n)$.*

Cameron later [3] suggested that Conjecture 1.1 is true with $c = 0.6$ and gave a lower bound construction:

$$S_0 = \{(x, y) \in [n]^2 \mid u \leq x + y \leq 2u - 1\},$$

which has maximum density 0.6 when $u = \lfloor \frac{4n+7}{5} \rfloor$. Recently, Elsholtz and Rackham settled Conjecture 1.1 in [7], proving that indeed

$$\mu([n]^2) = 0.6 + O(1/n).$$

In the same paper, Elsholtz and Rackham [7] raised the problem of classifying the sum-free sets whose size are close to the extremal value.

In this paper, we resolve this problem by showing that any sum-free subset $S \subseteq [n]^2$ of size at least $(\frac{3}{5} - o(1))n^2$ will have all its points in the region $\{(x, y) \in [n]^2 \mid \frac{4n}{5} - o(n) \leq x + y < \frac{8n}{5} + o(n)\}$.

Theorem 1.2. *For all $\gamma > 0$ there exists $\delta > 0$ and $n_0 \in \mathbb{N}$ such that the following holds for all $n > n_0$. If $S \subseteq [n]^2$ is sum-free with $|S| > (\frac{3}{5} - \delta)n^2$, then*

$$S \subseteq \{(x, y) \in [n]^2 \mid \frac{4n}{5} - \gamma n \leq x + y < \frac{8n}{5} + \gamma n\}.$$

This gives a satisfying answer to 2-dimension sum-free problem. The situation is, however, unclear for higher dimension. In particular, even the maximum density of a sum-free set in 3-dimension grid $[n]^3$ is unknown.

1.2 (p, q)-sum free sets

Given positive integers d, n and rational numbers p, q, a set $S \subseteq [n]^d$ is called (p, q)-*sum free* if it has no solution for the equation $px + qy = z$. As a generalization of sum free sets (i.e. (1,1)-sum free sets), the notion of (p, q)-sum free sets encapsulates many fundamental topics in combinatorial number theory. In particular, for $d = 1$, a $(\frac{1}{2}, \frac{1}{2})$-sum free set is precisely a set without 3-term arithmetic progression, which has received considerable attention in recent decades. Therefore, it is a natural question to determine the size of the largest (p, q)-sum free sets in $[n]^d$. Here one can similarly define

$$\mu_{[p,q]}([n]^d) := \max\left\{\frac{|S|}{n^d} \mid S \subseteq [n]^d \textit{ is } (p, q)\textit{-sum-free}\right\}.$$

By Roth's theorem [10], $\mu_{[1/2,1/2]}([n]) = o(1)$. See [2] for the best known upper bound for the size of a $(1/2, 1/2)$-sum free set. In [11, 12], instead of the form $x + y = z$, Ruzsa instigated the study of more general linear equations $a_1x_1 + \cdots + a_kx_k = b$. In particular, for more general cases when p, q are positive integers and $p \geq 2$, Hancock and Treglown [9] completely determined the value $\mu_{[p,q]}([n])$. For higher dimensional lattices, Choi and Kim [5] initiated the investigation of the form $x_1 + x_2 + \cdots + x_k = b$, where b is a prescribed points in $[n]^2$.

For 2-dimension (p, q)-sum free problem, we make the first attempt to determine $\mu_{[p,p]}([n]^2)$ for any integer p.

Theorem 1.3. *Let $p \in \mathbb{N}$ and $S \subseteq [n]^2$ be a (p, p)-sum free set. Then*

$$|S| \leq \Big(1 - \frac{2}{4p^2 + 1}\Big)n^2 + O(n).$$

We observe that the upper bound in Theorem 1.3 is optimal up to the error term $O(n)$, given by the following construction. For any positive integers p, q and positive real a, define $S = \{(x, y) \in [n]^2 \mid a < x + y < (p + q)a\}$. One can easily check that S is (p, q)-sum free with size

$$|S| = \left(1 - \frac{2}{(p + q)^2 + 1}\right) n^2 + O(n),$$

when $a = \frac{2(p+q)}{(p+q)^2+1}n$, corresponding to the stripe

$$S = \left\{(x, y) \in [n]^2 \mid \frac{2(p + q)}{(p + q)^2 + 1}n < x + y < \frac{2(p + q)^2}{(p + q)^2 + 1}n\right\}.$$

We conjecture that for all integers p and q, the above construction provides the maximum (p, q)-sum free set.

Conjecture 1.4. *Given positive integers p, q and n, where n is sufficiently large,*

$$\mu_{[p,q]}([n]^2) = \left(1 - \frac{2}{(p + q)^2 + 1}\right) + O(1/n).$$

Organization. The rest of the paper will be organized as follows. Section 2 includes some notation and tools needed. Section 3 is devoted to the proof of Theorem 1.2. The proof of Theorem 1.3 is given in Section 4.

2 Preliminaries

Given a convex polygon T in $\mathbb{R}^2_{[0,n]}$, denote by $\Lambda(T)$ the number of lattice points contained within T, and by $\|T\|$ the area of T with respect to the Lebesgue measure. Throughout the proof, we always use the following result of Skriganov [14] which relates the area of a shape with the number of lattice points contained inside it.

Lemma 2.1 (Theorem 5 in [14])**.** *If P is a convex polygon (a simple closed curve with finitely many points which are not part of a line segment) with at least one side having a rational normal, then*

$$\max_{a\in\mathbb{Z}^2} \Big|\big(\|tP + a\| - \Lambda(tP + a)\big)\Big| = o(t),$$

where tP denotes the dilation of P by $t > 0$.

This lemma implies that any convex polygon P, described above, satisfies that $\Lambda(P) = (1 + o(1))\|P\|$, which allows us to focus on the area $\|P\|$ instead of $\Lambda(P)$.

For two points $p_1, p_2 \in \mathbb{R}^2_{[0,n]}$, denote by $m(p_1, p_2)$ the gradient and by $c(p_2, p_2)$ the y-intercept of the line in $\mathbb{R}^2$ passing through p_1 and p_2.

Definition 2.2 (Upper boundary)**.** Given a set $A \subseteq \mathbb{R}^2_{[0,n]}$, the *upper boundary* of A is a set of points in A, denoted by ∂A, such that for each $p_1 \in \partial A$ there exists a point $p_2 \in A \setminus \{p_1\}$ with the following properties:

- $m(p_1, p_2) < 0$;
- Let $T = \{(x, y) \subseteq \mathbb{R}^2_{[0,n]} \mid y > m(p_1, p_2)x + c(p_1, p_2)\}$. Then $|A \cap T| = 0$.

Any two such points p_1, p_2 are said to be *adjoint*, and the line passing through two points that are adjoint is called an *upper boundary line*. The second condition above states that there is no point of A strictly above any upper boundary line.

The following lemma shows that if the upper boundary of a set A is empty, then A has a 'top right corner'.

Lemma 2.3 (Lemma 5.1 in [7])**.** *Suppose $A \subseteq \mathbb{R}^2_{[0,n]}$ such that $\partial A = \varnothing$. Then there is a point $(a, b) \in A$ such that $a \geq x$ and $b \geq y$ for all $(x, y) \in A$.*

We also need the concept of pairing sets, which will be frequently used throughout the proof.

Definition 2.4. Given a point $a \in \mathbb{R}^2_{[0,n]}$ and a set $P \subseteq \mathbb{R}^2_{[0,n]}$, we call P a *pairing set for a* if for any $x \in P$, we have $a - x \in P$.

The following lemma, which immediately follows from the definition of pairing set, guarantees that for any point in a sum-free set S, every pairing set for that point cannot intersect too much with S.

Lemma 2.5 (Lemma 3.4 in [7])**.** *Let S be a sum-free set in $[n]^2$. Then for any $a \in S$ and a pairing set P for a, we have $|P \cap S| \leq \frac{1}{2}\Lambda(P)$.*

The following lemma bounds the intersection of a set and its translate with a sum-free set.

Lemma 2.6. *Given two sets $S, T \subseteq [n]^2$, if S is sum-free, then for any $a \in S$, it holds that*

$$|S \cap (T \cup (a \pm T))| \leq \Lambda(T).$$

Proof. For each element $t \in T$ there is a corresponding element $a \pm t \in a \pm T$. Since $a \in S$, one can observe from sum-freeness that at most one of t and $a \pm t$ belongs to S. □

3 Proof of Theorem 1.2

We carry out the proof in a few steps. First, using Lagrange multiplier, we show that any almost maximum-size sum-free set S in $[n]^2$ has an upper boundary line that is close to the line $y + x = \frac{8n}{5}$, see Lemma 3.3. Then we show that there is a point (x^*, y^*) in S close to $(\frac{4n}{5}, \frac{4n}{5})$, see Lemma 3.4. Finally, using this point (x^*, y^*), we show in Section 3.3 that S has no point below the line $y+x = \frac{4n}{5} - o(n)$, which, together with the upper boundary line close to $y + x = \frac{8n}{5}$, implies that S must be close to the extremal stripe $\frac{4n}{5} \leq x + y \leq \frac{8n}{5}$.

Definition 3.1. A sum-free set $S \subseteq [n]^2$ with $\partial S \neq \varnothing$ is of *Type* 1 if there exists a point $p_1 = (x_1, y_1) \in \partial S$ with $x_1 \leq y_1$ and $x_1 y_1 \geq xy$ for all $(x, y) \in \partial S$, and a point $p_2 = (x_2, y_2)$ adjoint to p_1 satisfying the following conditions, where we simply write $m = m(p_1, p_2)$ and $c = c(p_1, p_2)$.

(1) $x_2 > x_1$, $y_2 < y_1$ and $m < -\frac{y_1}{x_1} \leq -1$;

(2) $c > n$ and $-c \leq nm$.

In addition, S is of *Type* 2 if there exist two adjoint points $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$ in ∂S satisfying the following conditions.

(1) $x_2 > x_1$, $y_2 < y_1$ and $-\frac{y_1}{x_1} \leq m \leq -\frac{y_2}{x_2}$;

(2) $y_2 \leq \frac{c}{2} \leq y_1$;

(3) $c > n$ and $-c < nm$.

For either type of the sum-free set, we call the upper boundary lines passing through p_1 and p_2 *typical*. Let

$$A = \{(x, y) \in \mathbb{R}^2_{[0,n]} \mid y > mx + c\}$$

with m and c given as above. Then A is a triangle in both cases.

For the Type 1 set S, we claim that the upper boundary line $y = mx + c$ satisfies $x_1 > \frac{n}{2}$. In fact, since $m < -\frac{y_1}{x_1}$ and $y_1 = mx_1 + c$, we have that $x_1 > \frac{c}{-2m} > \frac{n}{2}$ because $-c < nm$.

If S is of Type 2, then it is straightforward to check that the following two sets are nonempty.

$$T_1 = \left\{(x, y) \in \mathbb{R}^2_{[0,n]} \mid x \geq x_1, y - mx \leq \frac{c}{2}\right\},$$
$$T_2 = \left\{(x, y) \in \mathbb{R}^2_{[0,n]} \mid y \geq y_2, y - mx \leq \frac{c}{2}\right\}.$$

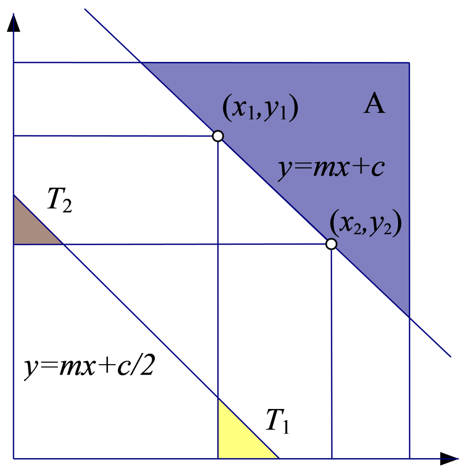

Figure 1: Type 2 S

The two types we defined above correspond to the only two cases in [7] that attain the maximum density $\frac{3}{5}$. We will use the following bounds for these two types of sum-free sets.

Lemma 3.2 ([7])**.** *Given a sum-free set $S \subseteq [n]^2$, if $|S| > 0.56n^2$, then either*

(1) *S is of Type 1 and $|S| \leq (n+1)^2 - \frac{1}{2}x_1 y_1 + \frac{(c+nm-n)^2}{2m}$, or*

(2) *S is of Type 2 and $|S| \leq (n+1)^2 + \frac{c^2}{8m} + \frac{1}{2m}(n - nm - c)^2$.*

3.1 Fixing an upper boundary line

Given constants ε and C, we call that a line L is *ε-close* to the line $x+y=C$ if the portion of L intersecting $\mathbb{R}^2_{[0,n]}$ lies entirely within the set $\{(x,y)\in\mathbb{R}^2_{[0,n]} \mid |x+y-C|<\varepsilon n\}$. Similarly, we call two point $p_1=(x_1,y_1)$ and $p_2=(x_2,y_2)$ are *ε-close* to each other if $|x_1-x_2|\le\varepsilon n$ and $|y_1-y_2|\le\varepsilon n$.

Lemma 3.3. *Given $\varepsilon>0$, there exist $\delta>0$ and $n_0\in\mathbb{N}$ such that the following holds for all $n>n_0$. If $S\subseteq[n]^2$ is sum-free and $|S|>(\frac{3}{5}-\delta)n^2$, then there is a typical upper boundary line for S which is ε-close to $x+y=\frac{8n}{5}$.*

Proof. Given $\varepsilon>0$, let $\delta=\frac{\varepsilon^2}{100}$ and n be sufficiently large with respect to ε. Let $S\subseteq[n]^2$ be a sum-free set with $|S|>(\frac{3}{5}-\frac{\varepsilon^2}{100})n^2$.

Suppose for contradiction that any upper boundary line $y=mx+c$ for S is not ε-close to $x+y=\frac{8n}{5}$. That is, either the y- or the x-intercept is far from where it should be:

$$\text{either}\quad |c-8n/5|>\varepsilon n \quad\text{or}\quad |c/m+8n/5|>\varepsilon n.$$

We shall derive in both cases a contradiction that $|S|\le(3/5-\varepsilon^2/100)n^2$.

Considering the typical upper boundary line $y=mx+c$ passing through p_1 and p_2 given in Definition 3.1, we will finish the case when the y-intercept is too far, that is, $|c-8n/5|>\varepsilon n$, whose proof will be divided into two cases depending on the type of S. The case when the x-intercept is too far (that is, $|c/m+8n/5|>\varepsilon n$) is similar and we omit the details.

Suppose first that S is of Type 1, then by Lemma 3.2(1), we have

$$|S|\le(n+1)^2-\frac{1}{2}\left(x_1(mx_1+c)-\frac{(c+mn-n)^2}{m}\right)=:f(x,m,c).$$

To simplify the presentation, we introduce a new variable η with $\eta\in(-\infty,-\varepsilon)\cup(\varepsilon,+\infty)$ and define

$$f_\eta:=\max\{f(x,m,c)\mid c-8n/5=\eta n\}.$$

Let $L:=f(x,m,c)-\lambda g$, where $g=c-8n/5-\eta n$. By solving $\frac{\partial L}{\partial x}=0$, $\frac{\partial L}{\partial m}=0$, $\frac{\partial L}{\partial c}=0$ and $\frac{\partial L}{\partial\lambda}=0$, we obtain $m=-\sqrt{1+2\eta+\frac{5\eta^2}{4}}$ and $x=\frac{\frac{4}{5}+\frac{\eta}{2}}{\sqrt{1+2\eta+\frac{5\eta^2}{4}}}n$, and thus the maximum value is

$$f_\eta=(8/5+\eta-\sqrt{1+2\eta+5\eta^2/4})n^2+O(n).$$

As η takes values over $(-\infty,-\varepsilon)\cup(\varepsilon,+\infty)$, we get

$$f_\eta\le(8/5+\varepsilon-\sqrt{1+2\varepsilon+5\varepsilon^2/4})n^2+O(n)\le(3/5-\varepsilon^2/100)n^2.$$

For the second case when S is of Type 2, by Lemma 3.2(2), we have:

$$|S|\le(n+1)^2+\frac{c^2}{8m}+\frac{(n-nm-c)^2}{2m}.$$

Using Lagrange multiplier again, we arrive at the same bound $(8/5+\varepsilon-\sqrt{1+2\varepsilon+5\varepsilon^2/4})n^2+O(n)\le(3/5-\varepsilon^2/100)n^2$ as desired. □

3.2 Top right corner

Lemma 3.4. *For any $\beta>0$, there exist $\delta>0$ and $n_0\in\mathbb{N}$ such that for all $n>n_0$, if $S\subseteq[n]^2$ is sum-free with $|S|>(\frac{3}{5}-\delta)n^2$, then there is a point $(x^*,y^*)\in S$ which is β-close to the point $(\frac{4n}{5},\frac{4n}{5})$.*

Proof. We first handle Type 1 sum-free sets. Given $\beta > 0$, we have constants $\delta = \delta_{3.3} > 0$ and $n_0 \in \mathbb{N}$ returned from Lemma 3.3 with $\varepsilon = \beta/6$. Let $S \subseteq [n]^2$ be a sum-free set of Type 1 with $|S| > (\frac{3}{5} - \delta)n^2$. Then Lemma 3.3 gives a typical upper boundary line $y = mx + c$ that is ε-close to $x + y = \frac{8n}{5}$ and let $p_1 = (x_1, y_1)$, $p_2 = (x_2, y_2)$ be the two points involved. Therefore, $|c - \frac{8n}{5}| < \varepsilon n$, $|x_1 + y_1 - \frac{8n}{5}| < \varepsilon n$. Consequently, by triangle inequality we have

$$|m + 1| = \frac{|x_1 + y_1 - c|}{x_1} < \frac{2\varepsilon n}{x_1} < 4\varepsilon,$$

where the last inequality follows since $x_1 > n/2$. Recall that $m \le -\frac{y_1}{x_1} \le -1$. Then we have that $|m + \frac{y_1}{x_1}| < 4\varepsilon$.

Using these facts we can write $m = -\frac{y_1}{x_1} - \gamma_1$ and $c = (\frac{8}{5} + \gamma_2)n$ for constants $0 \le \gamma_1 < 4\varepsilon$ and $|\gamma_2| < \varepsilon$. Using the equation $y_1 = mx_1 + c$, we obtain that $y_1 = \frac{4}{5}n + \frac{\gamma_2 n - \gamma_1 x_1}{2}$. As $x_1 \le n$, by triangle inequality, we have

$$\left| y_1 - \frac{4n}{5} \right| < \frac{5\varepsilon n}{2} < \beta n.$$

Moreover, since $-\frac{y_1}{x_1} \ge m > -1 - 4\varepsilon$ and $x_1 \le y_1$, we can easily obtain that $|x_1 - \frac{4n}{5}| < 6\varepsilon n = \beta n$. So (x_1, y_1) is β-close to the point $(\frac{4n}{5}, \frac{4n}{5})$ as desired.

Let us turn to Type 2 sum-free set. Now, given $\beta > 0$, choose positive constants ε, δ with $\delta \ll \varepsilon \ll \beta$. Let S be a sum-free set of Type 2 with $|S| > (\frac{3}{5} - \delta)n^2$. Then applying Lemma 3.3 with $\sqrt{2}\varepsilon$ playing the role of ε gives a typical upper boundary line $y = mx + c$ passing through (x_1, y_1) and (x_2, y_2) which is $\sqrt{2}\varepsilon$-close to $y + x = \frac{8n}{5}$. This implies that the line $y = mx + \frac{c}{2}$ is $\frac{\varepsilon}{\sqrt{2}}$-close to $x + y = \frac{4n}{5}$. We may assume for contradiction that S has no points in the region

$$T_\beta = \left\{ (x, y) \in \mathbb{R}^2_{[0,n]} \mid x, y \ge \frac{4n}{5} - \beta n \,,\ \ y + x \le \frac{8n}{5} \right\}.$$

Redefine the regions as follows:

$$A = \left\{ (x, y) \in \mathbb{R}^2_{[0,n]} \mid y + x \ge \frac{8n}{5} + \sqrt{2}\varepsilon n \right\},$$

$$T_1 = \left\{ (x, y) \in \mathbb{R}^2_{[0,n]} \mid y + x \le \frac{4n}{5} - \frac{\varepsilon n}{\sqrt{2}},\ x \ge x_1 \right\},$$

$$T_2 = \left\{ (x, y) \in \mathbb{R}^2_{[0,n]} \mid y + x \le \frac{4n}{5} - \frac{\varepsilon n}{\sqrt{2}},\ y \ge y_2 \right\}.$$

Note that

$$T_1 + T_2 = \left\{ (x, y) \in \mathbb{R}^2_{[0,n]} \mid y + x \le \frac{8n}{5} - \sqrt{2}\varepsilon n,\ y_2 \le y,\ x_1 \le x \right\}.$$

We now proceed by considering the areas which may be excluded from S. Firstly, we show that S has two points in T_1 that are far apart.

Claim 3.5. *There are two points in $T_1 \cap S$ which are at least βn far apart.*

Proof of claim. If this is not true, then there are less than $\beta^2 n^2$ points of S in T_1, given by the number of points in a square around a circle of diam βn in T_1. Since $\|T_\beta\| = \frac{1}{2}(2\beta)^2 n^2 = 2\beta^2 n^2$, we then use the pairing set P_1 for (x_1, y_1) and thus

$$\begin{aligned}
|S| &\le n^2 - \Lambda(A) - \frac{1}{2}\Lambda(P_1) - \Lambda(T_1) + \beta^2 n^2 - \Lambda(T_\beta) \\
&= n^2 - \|A\| - \frac{1}{2}\|P_1\| - \|T_1\| + \beta^2 n^2 - \|T_\beta\| + O(n) \\
&= n^2 - \frac{1}{2}\left(\frac{2}{5} - \sqrt{2}\varepsilon\right)^2 n^2 - \frac{1}{2}x_1 y_1 - \frac{1}{2}\left(\frac{4}{5} - \frac{\varepsilon}{\sqrt{2}} - \frac{x_1}{n}\right)^2 n^2 + \beta^2 n^2 - 2\beta^2 n^2 + O(n)
\end{aligned}$$

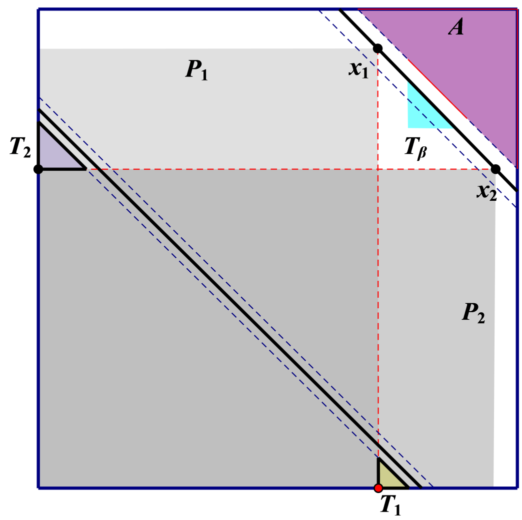

Figure 2: S is Type 2

It is easy to see this is maximized when y_1 is minimal and $x_1 + y_1 = \frac{8n}{5} - \sqrt{2}\varepsilon n$. Then

$$|S| \leq \left(\frac{3}{5} - \beta^2 + 10\varepsilon\right) n^2.$$

Therefore, we reach a contradiction by the fact that $\delta \ll \varepsilon \ll \beta$. ■

By Claim 3.5, we let t' and t'' be two points in T_1 with distance greater than βn, and let

$$T_2' := t' + T_2 \quad \text{and} \quad T_2'' := t'' + T_2.$$

Claim 3.6. *$\Lambda(T_\beta \setminus T_2'), \Lambda(T_\beta \setminus T_2'') < \frac{\beta^2}{L} n^2$, where $L := \frac{4}{5\sqrt{3}-6}$.*

Proof of claim. Suppose to the contrary that either $\Lambda(T_\beta \setminus T_2') \geq \frac{\beta^2}{L} n^2$ or $\Lambda(T_\beta \setminus T_2'') \geq \frac{\beta^2}{L} n^2$, and by symmetry we may assume the first inequality holds. Considering the pairing set P_2 for (x_2, y_2) and T_2 paired with T_2', we can obtain from Lemmas 2.5 and 2.6 that

$$\begin{aligned} |S| &\leq n^2 - \Lambda(A) - \frac{1}{2}\Lambda(P_2) - \Lambda(T_2) - \Lambda(T_\beta \setminus T_2') \\ &\leq \left(\frac{3}{5} - \frac{\beta^2}{L} + O(\varepsilon)\right) n^2, \end{aligned}$$

which once again gives a contradiction as $\delta \ll \beta$. ■

In the rest of the proof, we shall find a partition $T_2 = T_{2,1} \cup T_{2,2}$ into two regions such that theirs corresponding translates $T_{2,1}' = t' + T_{2,1}$ and $T_{2,2}'' = t'' + T_{2,2}$ are distantly separated in $T_1 + T_2$, which provides a significant portion of points in $T_\beta \setminus (T_{2,1}' \cup T_{2,2}'')$ that are to be excluded from S.

Write $t' = (x', y')$ and $t'' = (x'', y'')$. By Claim 3.6, we can find that the two points $t' + (0, y_2)$ and $t'' + (0, y_2)$ belong to the region $\{(x, y) \in [n]^2 \mid x, y \leq \frac{4n}{5}\}$. We may assume $x' + y' \geq x'' + y''$ and let $d := x' + y' - (x'' + y'')$. It is easy to see that d is the difference between the corresponding y-intercepts of the red diagonal and the blue diagonal in Figure 3. By the symmetry of all the shapes involved, we can further assume that $x' \geq x''$ and $x' - x'' \geq y'' - y'$.

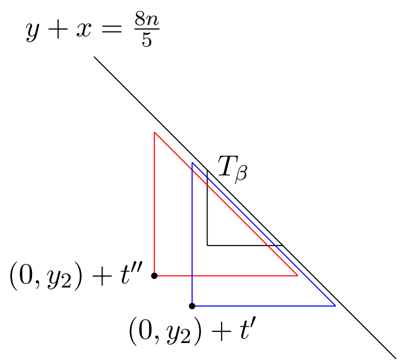

Figure 3: The red triangle represents T_2'', the blue one represents T_2' and the black one represents T_β.

Claim 3.7. *$d \le \left(2 - 2\sqrt{\frac{2L-1}{2L}}\right)\beta n = \frac{\sqrt{3}-1}{2}\beta n$.*

Proof of claim. It is easy to see the region $T_\beta \cap T_2''$ is a triangle similar to T_β. Note that the area of $T_\beta \setminus T_2''$ is at least

$$\frac{1}{2}(2\beta n)^2 - \frac{1}{2}(2\beta n - d)^2 = \left(2\beta n - \frac{d}{2}\right)d.$$

By Claim 3.6, we have that $\left(2\beta n - \frac{d}{2}\right) d \le \frac{\beta^2 n^2}{L}$, which yields the bound on d as desired. ■

Define points

$$X_1 = (x', y' + y_2), \quad X_2 = (x'', y'' + y_2) \quad \text{and} \quad X_3 = (x', y'' + y_2).$$

Let P_1X_1 and P_2X_2 be line segments which are parallel to PX_3 (see Figure 4). Construct a line passing through $(0, y_2)$ which is also parallel to the line segments PX_3, where $P = (\frac{4n}{5}, \frac{4n}{5})$. Such a line separates T_2 into two parts, and we denote by $T_{2,2}$ the part above the line and $T_{2,1}$ for the rest.

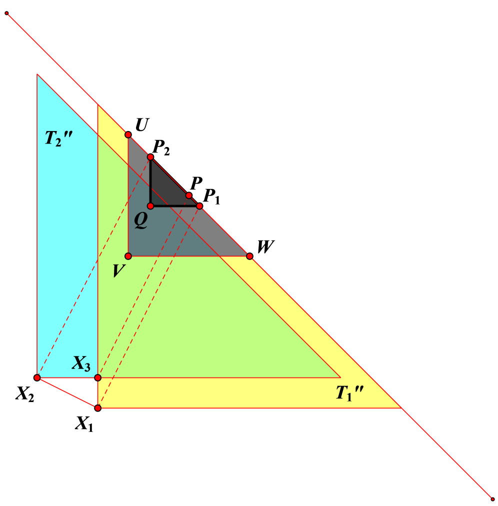

Figure 4: The triangle UVW represents T_β, in which $P = (\frac{4n}{5}, \frac{4n}{5})$ is the median point for the line segment UW.

Claim 3.8. *There exists a triangle $T_{\beta,1} \subseteq T_\beta$ similar to T_β such that $T_{\beta,1}$ does not intersect with either of the regions $T_{2,1}+t'$ or $T_{2,2}+t''$ and $\Lambda(T_{\beta,1}) \geq \frac{\beta^2 n^2}{8}$.*

Proof of claim. Recall that $s := y''-y'$ is positive. Since t' and t'' are of distance at least βn far apart, that is, $(x'-x'')^2+(y'-y'')^2 = (s+d)^2+s^2 \geq \beta^2 n^2$, together with Claim 3.7, we obtain that $s \geq \frac{\beta n}{2}$. Thus, the segment P_1P_2 has length at least $\frac{\sqrt{2}}{2}\beta n$. Let $T_{\beta,1}$ be the rectangle triangle P_1P_2Q with diagonal line P_1P_2. Then $T_{\beta,1}$ has area at least $\frac{\beta^2n^2}{8}$ and does not intersect either of the regions $T_{2,1}+t'$ or $T_{2,2}+t''$. ■

As aforementioned, now we are ready to finish the proof. Applying Lemma 2.6 to $T_{2,1}, T_{2,2}$ and their translates $T_{2,1}+t'$, $T_{2,2}+t''$, we obtain that

$$\begin{aligned}
|S| &\leq n^2 - \Lambda(A) - \frac{1}{2}\Lambda(P_2) - \Lambda(T_2) - \Lambda(T_{\beta,1}) \\
&= n^2 - \|A\| - \frac{1}{2}\|P_2\| - \|T_2\| - \|T_{\beta,1}\| + O(n) \\
&\leq n^2 - \frac{1}{2}\left(\frac{2}{5} - \sqrt{2}\varepsilon\right)^2 n^2 - \frac{1}{2}x_2y_2 - \frac{1}{2}\left(\frac{4}{5} - \frac{\varepsilon}{\sqrt{2}} - \frac{x_2}{n}\right)^2 n^2 - \frac{\beta^2 n^2}{8} + O(n).
\end{aligned}$$

The right-hand side above is maximized when y_2 is minimal and $x_2+y_2 = \frac{8n}{5} - \sqrt{2}\varepsilon n$. Thus,

$$|S| \leq \left(\frac{3}{5} - \frac{\beta^2}{8} + O(\varepsilon)\right) n^2,$$

a final contradiction. □

3.3 Putting things together

We are now ready to prove our main result, knowing that any almost maximum sum-free set contains an upper boundary line $o(1)$-close to $y+x=\frac{8n}{5}$ and a point $o(1)$-close to $(\frac{4n}{5},\frac{4n}{5})$.

Proof of Theorem 1.2. Given $\gamma>0$, choose $\delta \ll \varepsilon \ll \beta \ll \gamma$. Let $S \subseteq [n]^2$ be a sum-free set of size at least $(3/5-\delta)n^2$. Then by Lemma 3.3, S has a typical upper boundary line $y = mx+c$ which is ε-close to $y+x=\frac{8n}{5}$. Now it suffices to show that S has no point below the line $x+y=\frac{4n}{5}-\gamma n$ (see the red line in Figure 5).

Note that Lemma 3.4 ensures the existence of a point (x_1,y_1) of S that is β-close to $(\frac{4n}{5},\frac{4n}{5})$. Suppose to the contrary that $p_0=(x_0,y_0)\in S$ is such a point below the line $x+y=\frac{4n}{5}-\gamma n$, and without loss of generality we may assume that $y_0 \geq x_0$.

Let

$$A =: \left\{(x,y)\in[n]^2 \mid y+x > \frac{8n}{5}+\varepsilon n\right\}.$$

Considering the pairing set $P := \{(x,y)\in[n]^2 \mid x\leq x_1, y\leq y_1\}$ for (x_1,y_1), there are at most

$$n^2 - \Lambda(A) - \frac{1}{2}\Lambda(P) \leq \left(\frac{3}{5} + \left(\frac{2}{5}-\frac{\varepsilon}{2}\right)\varepsilon + \left(\frac{4}{5}-\frac{\beta}{2}\right)\beta\right) n^2 + O(n) \tag{1}$$

points which may be included in S; and all these points are below the line below the line $x+y=\frac{8n}{5}+\varepsilon n$. Then, writing

$$D_1 := \left\{(x,y)\in[n]^2 \mid y > \frac{4n}{5}+\beta n,\ y+x < \frac{8n}{5}-\varepsilon n\right\}$$

and

$$D_2 := \left\{(x,y) \in [n]^2 \mid x > \frac{4n}{5} + \beta n,\ y + x < \frac{8n}{5} - \varepsilon n\right\},$$

it follows from the assumption $|S| \geq (3/5 - \delta)n^2$ and (1) that

$$\frac{1}{n^2}|(D_1 \cup D_2) \setminus S| \leq \delta + \left(\frac{2}{5} - \frac{\varepsilon}{2}\right)\varepsilon + \left(\frac{4}{5} - \frac{\beta}{2}\right)\beta =: \upsilon(\delta, \varepsilon, \beta). \tag{2}$$

Note that we can choose $\delta, \varepsilon, \beta$ small enough such that $\upsilon(\delta, \varepsilon, \beta) = o(\gamma^2)$. In the remaining proof, we shall find in $D_1 \cup D_2$ (or its translate) a relatively large subset of lattice points which are to be excluded from S, yielding a contradiction.

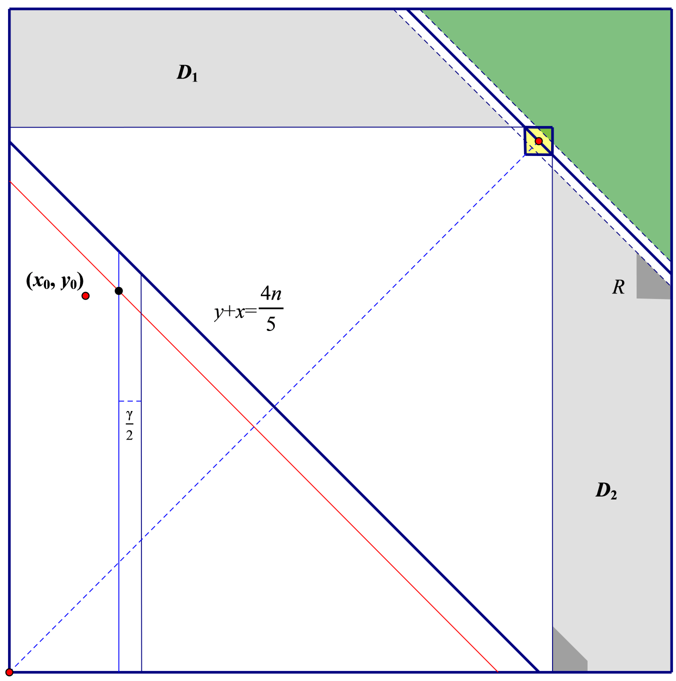

Figure 5: The case when $x_0 < \frac{n}{5} - \frac{\gamma}{2}n$: The two grey regions R and its translate $R - (x_0, y_0)$ form a pairing, which excludes from the sum-free set S the amount of points which fit in one of the regions .

First assume that p_0 is such that $x_0 < \frac{n}{5} - \frac{\gamma}{2}n$. Then the region $(D_1 \cup D_2) + p_0$ intersects $D_1 \cup D_2$ on a set of lattice points, denoted by R. Since $R, R - p_0 \subseteq D_1 \cup D_2$, applying Lemma 2.6 with $a = p_0$ and $T = R - p_0$ gives that $|(R \cup (R - p_0)) \cap S| \leq |R|$, and thus

$$|(D_1 \cup D_2) \setminus S| \geq |(R \cup (R - p_0)) \setminus S| \geq |R \cup (R - p_0)| - |R|.$$

It is easy to observe that $|R \cup (R - p_0)| - |R|$ is minimized when p_0 is close to the point $(\frac{n}{5} - \frac{\gamma n}{2}, \frac{3n}{5} - \frac{\gamma n}{2})$, yielding a area of size at least $\left(\frac{3}{8}\gamma^2 - \frac{\gamma - 2\beta}{4}(2\varepsilon - \beta)\right)n^2 + O(n)$ (See Figure 5). Thus $|(D_1 \cup D_2) \setminus S| \geq \left(\frac{3}{8}\gamma^2 - \frac{\gamma-2\beta}{4}(2\varepsilon - \beta)\right)n^2 + O(n) > \upsilon(\delta, \varepsilon, \beta)n^2$, a contradiction to (2).

Now it remains to consider the case when p_0 satisfies $x_0 \geq \frac{n}{5} - \frac{\gamma n}{2}$. We consider the overlap of $(D_1 \cup D_2) - p_0$ with $(x_1, y_1) - ((D_1 \cup D_2) - p_0)$ and denote by $\mathcal{O}$ the set of lattice points in the overlap (see Figure 6). Let

$$D := ((D_1 \cup D_2) - p_0) \setminus (D_1 \cup D_2).$$

Then it is easy to verify that $\mathcal{O} \subseteq D$. Note that by Lemma 2.6 with $a = p_0$ and $T = D_1 \cup D_2$, one have that

$$|(D \cup D_1 \cup D_2) \cap S| \le |D_1 \cup D_2|.$$

Then, using (2), we have

$$\begin{aligned}|\mathcal{O} \cap S| \le |D \cap S| &= |(D \cup D_1 \cup D_2) \cap S| - |(D_1 \cup D_2) \cap S| \\ &\le |(D_1 \cup D_2) \setminus S| \le \upsilon(\delta, \varepsilon, \beta) n^2.\end{aligned}$$

Moreover, by definition we know that $(x_1, y_1) - \mathcal{O} \subseteq \mathcal{O}$, that is, $\mathcal{O}$ (and also $P \setminus \mathcal{O}$) is a pairing set for (x_1, y_1). It follows from Lemma 2.5 that

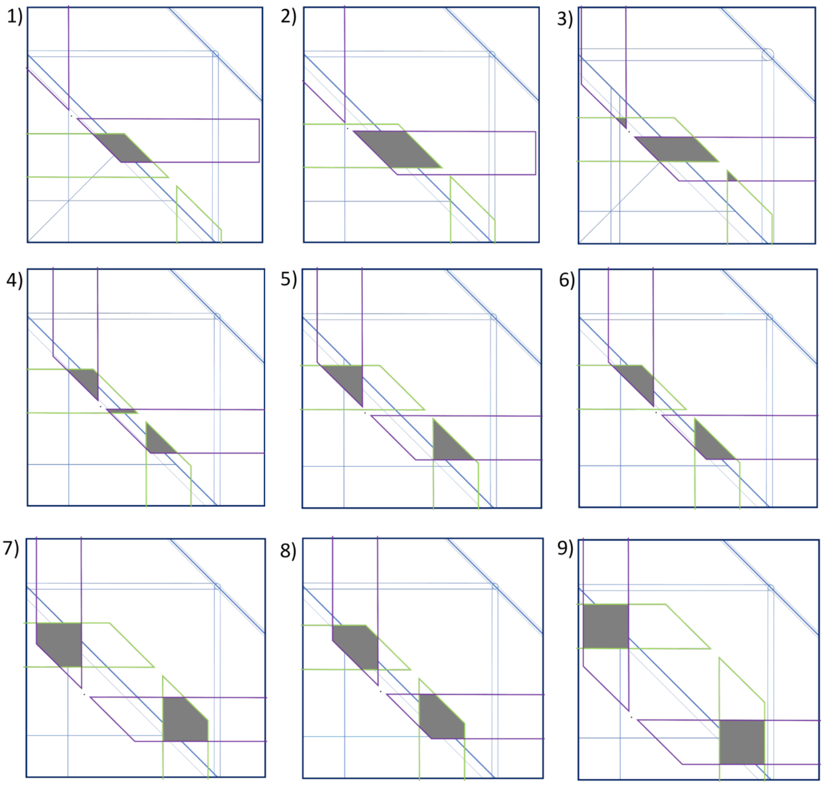

Figure 6: All possible shapes of $\mathcal{O}$

$$\begin{aligned}|S| &\le n^2 - \Lambda(A) - \frac{1}{2}\Lambda(P \setminus \mathcal{O}) - (|\mathcal{O}| - |\mathcal{O} \cap S|) \\ &\le n^2 - \|A\| - \frac{1}{2}\|P\| - \frac{1}{2}|\mathcal{O}| + \upsilon(\delta, \varepsilon, \beta) n^2 \\ &= n^2 - \frac{1}{2}\Big(\frac{2n}{5} - \varepsilon n\Big)^2 - \frac{1}{2}\Big(\frac{4n}{5} - \beta n\Big)^2 - \frac{1}{2}|\mathcal{O}| + \upsilon(\delta, \varepsilon, \beta) n^2 \\ &= \frac{3}{5}n^2 + o(\gamma^2) n^2 - \frac{1}{2}|\mathcal{O}|.\end{aligned}$$

Therefore, it suffices to show that $|\mathcal{O}| = \Omega(\gamma^2) n^2$, and in the remaining proof we shall verify this by considering all possible shapes of $\mathcal{O}$.

Since (x_1, y_1) is β-close to $(\frac{4n}{5}, \frac{4n}{5})$ and $\beta = o(\gamma^2)$, we may further assume that $(x_1, y_1) = (\frac{4n}{5}, \frac{4n}{5})$ in order not to cluster the presentation. We list in Figure 6 all possible shapes of the overlap $\mathcal{O}$, which originate from the location of the point (x_0, y_0) (see Figure 7). In particular, the area of the overlap in each of these cases is given as follows:

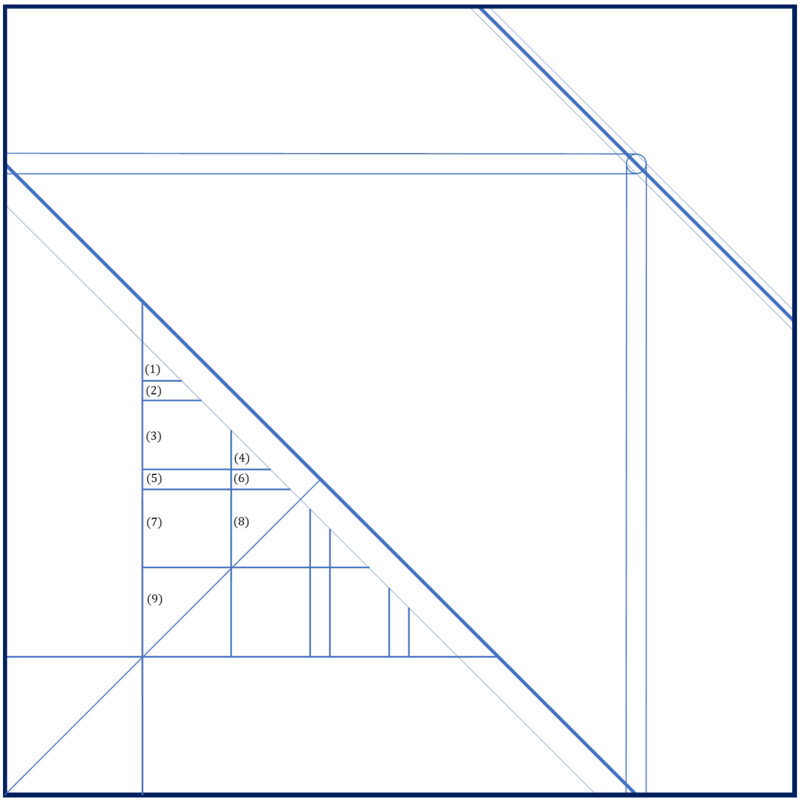

Figure 7: each numbered region will produce a unique shape of the overlap.

1. $4\left(\frac{3n}{5} - y_0\right)\left(\frac{4n}{5} - y_0 - x_0 - \varepsilon n\right)$, where $\frac{n}{2} + \frac{\beta n}{2} \le y_0 \le \frac{3n}{5} - \frac{\gamma n}{2}$.
2. $4\left(y_0 - \frac{2n}{5} - \beta n\right)\left(\frac{4n}{5} - y_0 - x_0 - \varepsilon n\right)$, where $\frac{n}{2} - \frac{(\beta+\varepsilon)n}{2} \le y_0 \le \frac{n}{2} + \frac{\beta n}{2}$.
3. $(n - 2y_0 - \beta n - \varepsilon n)^2 + 4\left(y_0 - \frac{2n}{5} - \beta n\right)\left(\frac{4n}{5} - y_0 - x_0 - \varepsilon n\right)$, where $\frac{2n}{5} + \beta n \le y_0 \le \frac{n}{2} - \frac{(\beta+\varepsilon)n}{2}$.
4. $4\left(\frac{4n}{5} - y_0 - x_0 - \varepsilon n\right)\left(x_0 - \frac{n}{5} - 2\beta n\right)$, where $x_0 \ge \frac{3n}{10} + \frac{(\beta-\varepsilon)n}{2}$.
5. $(n - 2y_0 - \beta n - \varepsilon n)^2$
6. $4\left(\frac{4n}{5} - y_0 - x_0 - \varepsilon n\right)\left(\frac{n}{5} + x_0 - y_0 - \beta n\right)$, where $x_0 \ge \frac{3n}{10} + \frac{(\beta-\varepsilon)n}{2}, y_0 \in [\frac{2n}{5} - \frac{\varepsilon n}{2}, \frac{2n}{5} + \beta n]$.
7. $2\left(\frac{n}{5} - \beta n\right)^2 - \left(2y_0 - \frac{3n}{5} + \varepsilon n - \beta n\right)^2$
8. $2\left(\frac{n}{5} - \beta n\right)^2 - \left(2y_0 - \frac{3n}{5} + \varepsilon n - \beta n\right)^2 - \left(2x_0 - \frac{3n}{5} + \varepsilon n - \beta n\right)^2$, where
$$\frac{3n}{10} + \frac{(\beta-\varepsilon)n}{2} \le x_0 \le y_0 \le \frac{2n}{5} - \frac{\varepsilon n}{2}.$$
9. $2\left(\frac{n}{5} - \beta n\right)^2$.

It is obvious that for the regions 5, 7 and 9, the area of the overlap has size $\Omega(\gamma^2)n^2$. The only regions which interest us are the ones bordering the line $y + x = \frac{4n}{5} - \gamma n$. Moreover, the regions in question are 1, 2, 3, 4, 6 and 8. Among them, the minimum overlap is achieved in region 1 by letting $(x_0, y_0) = (\frac{n}{5} - \frac{\gamma n}{2}, \frac{3n}{5} - \frac{\gamma n}{2})$, which yields a value of $|\mathcal{O}| \ge 2\gamma(\gamma - \varepsilon)n^2$ as desired. This completes the proof of Theorem 1.2. $\square$

4 Proof of Theorem 1.3

In this section we investigate the maximum size of a (p,p)-sum free set S. To simplify the presentation, we write p-sum free for (p,p)-sum free. Our proof builds on the techniques developed in the work of Elsholtz and Rackham [7]. We need a variant notion of pairing set as follows.

Definition 4.1. For any $(a_1,a_2) \in \mathbb{R}^2_{[0,n]}$, $P \subseteq \mathbb{R}^2_{[0,n]}$ is a p-pairing set for (a_1,a_2) if, for any $(x_1,x_2) \in P$, we have $(\frac{a_1}{p} - x_1, \frac{a_2}{p} - x_2) \in P$.

Analogous to Lemmas 2.5 and 2.6, the following lemma guarantees that for any point $a \in S$ and its p-pairing set P, at least half of the points in P are excluded from S. Similar statement also holds when we consider a set and its translate dilated by p. We omit the proof.

Lemma 4.2. *Let $S \subseteq [n]^2$ be a p-sum free set.*

(1) *If P is a p-pairing set for some $a \in S$, then we have $|S \cap P| \leq \frac{1}{2}\Lambda(P)$.*

(2) *If $T \subseteq \mathbb{R}^2_{[0,n]}$ and $a \in S$, then $|S \cap (p(a+T) \cup T)| \leq \Lambda(T)$.*

Proof of Theorem 1.3. Let $S \subseteq [n]^2$ be a p-sum free set. Our goal is to show that $|S| \leq \left(1 - \frac{2}{4p^2+1}\right) n^2 + O(n)$ for $p \geq 2$. We may neglect any boundary effects as they give error terms $O(n)$ for the size of S, which will be omitted so as to ease the presentation. We consider cases depending on the placement of upper boundary lines.

Case 1: $|\partial(S)| \leq 1$. As vertices in the upper boundary come in (adjoint) pairs, we see that in this case $\partial(S) = \varnothing$, and thus Lemma 2.3 ensures the existence a point $p_1 = (x_1,y_1) \in S$ such that $x_1 \geq x$ and $y_1 \geq y$ for all $(x,y) \in S$. Let $P := \{(x,y) \mid 0 \leq x \leq \frac{x_1}{p}, 0 \leq y \leq \frac{y_1}{p}\}$. Then P is a p-pairing set for p_1 and thus by Lemma 4.2, we have that

$$\begin{aligned}|S| &\leq (n+1)^2 - (n-x_1)n - (n-y_1)x_1 - \frac{1}{2}\Lambda(P)\\ &= \left(1 - \frac{1}{2p^2}\right) x_1y_1 + O(n) \leq \left(1 - \frac{1}{2p^2}\right) n^2 + O(n) < \left(1 - \frac{2}{4p^2+1}\right) n^2.\end{aligned}$$

Case 2: $|\partial(S)| \geq 2$ and for every two points $p_1 = (x_1,y_1), p_2 = (x_2,y_2)$ that are adjoint in $\partial(S)$ with $x_1 < x_2$ and $y_1 > y_2$, we have either $m(p_1,p_2) > -\frac{y_2}{x_2}$ or $m(p_1,p_2) < -\frac{y_1}{x_1}$.

In this case, we choose $p_1 = (x_1,y_1) \in \partial(S)$ such that $x_1y_1 \geq xy$ holds for every $(x,y) \in \partial(S)$ and $P_1 := \{(x,y) \mid 0 \leq x \leq \frac{x_1}{p}, 0 \leq y \leq \frac{y_1}{p}\}$. By symmetry, we may further assume that $y_1 \geq x_1$. If there does not exist $p_2 = (x_2,y_2) \in \partial(S)$ adjoint to p_1 with $x_2 > x_1$ and $y_2 < y_1$, then by Lemma 4.2 and that $y_1 \geq x_1$, we have

$$|S| \leq n^2 - (n-x_1)n - \frac{1}{2}\Lambda(P_1) \leq nx_1 - \frac{x_1^2}{2p^2} \leq \left(1 - \frac{1}{2p^2}\right) n^2.$$

Thus, we may assume that there exists $p_2 = (x_2,y_2) \in \partial(S)$ adjoint to p_1 with $x_2 > x_1$ and $y_2 < y_1$. Let $L : y = mx + c$ be the line passing through p_1, p_2 and define

$$A = \{(x,y) \in \mathbb{R}^2_{[0,n]} \mid y > mx + c\}.$$

We claim that $m < -\frac{y_1}{x_1} \leq -1$. Indeed, by the assumption of Case 2, assume for contradiction that $m > -\frac{y_2}{x_2}$, then

$$x_2y_2 = x_2(y_1 + m(x_2 - x_1)) \geq x_2y_1 - y_2(x_2 - x_1) = x_1y_1 + (y_1 - y_2)(x_2 - x_1) > x_1y_1,$$

contrary to the choice of p_1.

We split into two subcases depending on the x- and y-intercept of L. Note first that, if $c \le n$, then we have $-\frac{c}{m} \le n$ because $m \le -1$, and so $|S| < \frac{1}{2}n^2$ as $A \cap S = \varnothing$.

(I). If $c > n$ and $-\frac{c}{m} \le n$, then

$$|S| \le n^2 - \Lambda(A) - \frac{1}{2}\Lambda(P_1) = \frac{n}{m}\left(\frac{n}{2} - c\right) - \frac{1}{2p^2}x_1y_1 = \frac{n}{m}\left(\frac{n}{2} - y_1\right) + x_1 n - \frac{1}{2p^2}x_1y_1.$$

Now if $y_1 \le \frac{n}{2}$, then as $m < -1$ and $x_1 \le y_1 \le \frac{n}{2}$, we observe that $|S| \le x_1 n \le \frac{1}{2}n^2$. We may then assume $y_1 > \frac{n}{2}$.

If $x_1 < \frac{n}{2}$, then by the assumption that $m < -\frac{y_1}{x_1}$, we have

$$|S| \le \frac{nx_1}{y_1}\left(y_1 - \frac{n}{2}\right) + x_1 n - \frac{1}{2p^2}x_1y_1 \le \left(2n - \frac{n^2}{2y_1} - \frac{y_1}{2p^2}\right)\frac{n}{2} \le \left(1 - \frac{1}{2p}\right)n^2.$$

where the last inequality follows from $\frac{n^2}{2y_1} + \frac{y_1}{2p^2} \ge 2\sqrt{\frac{n^2}{2y_1}\frac{y_1}{2p^2}} = \frac{n}{p}$.

Assume then $x_1 \ge \frac{n}{2}$. Note that as $-\frac{c}{m} \le n$, the slope of L is smaller than the slope of the line passing through p_1 and $(n, 0)$, and so $m \le \frac{-y_1}{n - x_1}$. Thus, we have

$$\begin{aligned}|S| &\le \frac{n(n-x_1)}{y_1}\left(y_1 - \frac{n}{2}\right) + x_1 n - \frac{1}{2p^2}x_1y_1 \le n^2 - \left(\frac{n-x_1}{2n}n^2 + \frac{x_1y_1}{2p^2}\right)\\ &= \frac{n^2}{2} + \left(\frac{n}{2} - \frac{y_1}{2p^2}\right)x_1 \le \frac{n^2}{2} + \left(\frac{n}{2} - \frac{y_1}{2p^2}\right)y_1 \le \left(1 - \frac{1}{2p^2}\right)n^2,\end{aligned}$$

where the second last inequality follows since $x_1 \le y_1$ and the last one follows from $p \ge 2$.

(II). If $c > n$ and $-\frac{c}{m} > n$, then A is a triangle and thus

$$\begin{aligned}|S| &\le n^2 - \Lambda(A) - \frac{1}{2}\Lambda(P_1)\\ &= n^2 + \frac{(n-y_1)^2}{2m} + \frac{m(n-x_1)^2}{2} - (n-x_1)(n-y_1) - \frac{x_1y_1}{2p^2}.\end{aligned}$$

The right-hand side above is increasing when $m \le -\frac{n-y_1}{n-x_1}$. Since $\frac{n-y_1}{n-x_1} \le \frac{y_1}{x_1} \le -m$, it follows that

$$\begin{aligned}|S| &\le n^2 - \frac{(n-y_1)^2}{\frac{2y_1}{x_1}} - \frac{y_1(n-x_1)^2}{2x_1} - (n-x_1)(n-y_1) - \frac{x_1y_1}{2p^2}\\ &\le 2(x_1+y_1)n - n^2 - \left(2 + \frac{1}{2p^2}\right)x_1y_1,\end{aligned}$$

where the right-hand side of the last inequality is maximized when $x_1 = y_1 = \frac{4p^2}{4p^2+1}n$, and thus $|S| \le \left(1 - \frac{2}{4p^2+1}\right)n^2$.

Case 3: There exist $p_1 = (x_1, y_1), p_2 = (x_2, y_2)$ adjoint in $\partial(S)$ such that $x_1 < x_2$, $y_1 > y_2$ and $-\frac{y_1}{x_1} \le m(p_1, p_2) \le -\frac{y_2}{x_2}$.

Let P_i be a p-pairing set for each p_i with $i \in [2]$, that is, $P_i := \{(x, y) \mid 0 \le x \le \frac{x_i}{p}, 0 \le y \le \frac{y_i}{p}\}$. Set $A = \{(x, y) \in \mathbb{R}^2_{[0,n]} \mid y > mx + c\}$. Since $m \le -\frac{y_2}{x_2}$ and $y_2 = mx_2 + c$, we have that $y_2 \le \frac{c}{2}$. Similarly, by the condition $m \ge -\frac{y_1}{x_1}$, we have that $y_1 \ge \frac{c}{2}$.

Define

$$T_1 = \{(x, y) \in \mathbb{R}^2_{[0,n]} \mid x \ge \frac{x_1}{p}, y \le mx + \frac{c}{2p}\},$$

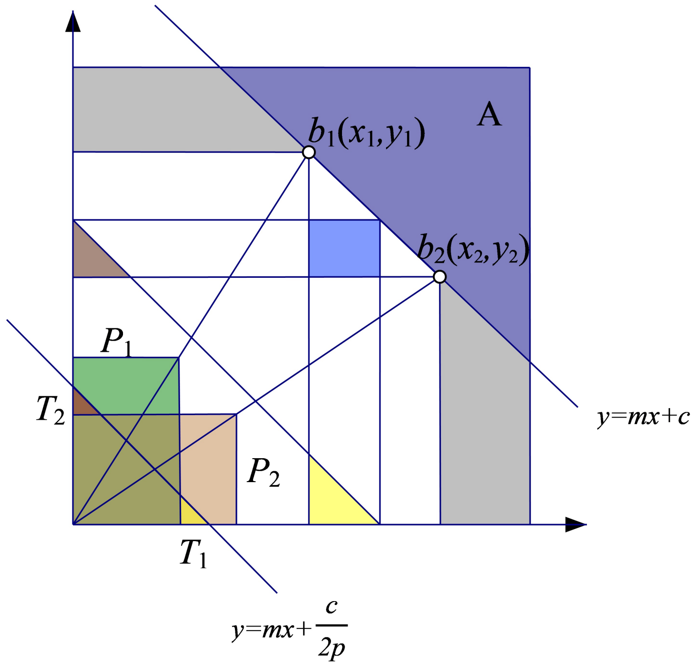

Figure 8: Nonempty T_1, T_2.

and

$$T_2 = \{(x,y) \in \mathbb{R}^2_{[0,n]} \mid y \geq \frac{y_2}{p}, y \leq mx + \frac{c}{2p}\}.$$

We claim that $T_1, T_2 \neq \varnothing$. These amount to proving $-\frac{c}{2mp} \geq \frac{x_1}{p}$ and $\frac{c}{2p} \geq \frac{y_2}{p}$, which in turn follows from the fact that $y_2 \leq \frac{c}{2} \leq y_1$, see Figure 8.

If $T_1 \cap S = \varnothing$, then a short calculation shows

$$|S| \leq n^2 - \Lambda(T_1) - \Lambda(A) - \frac{1}{2}\Lambda(P_1) \leq n^2 + \frac{c^2}{8p^2m} - \|A\|.$$

If $T_1 \cap S \neq \varnothing$, then take a point $a \in T_1 \cap S$, then one can check that $p(a + T_2) \cap T_2 = \varnothing$. By Lemma 4.2, we have $|S \cap (p(a + T_2) \cup T_2)| \leq \Lambda(T_2)$. By the definition of T_2, any point $(x, y) \in p(a + T_2)$ satisfies that $y \leq mx + c$ and $x \geq x_1, y \geq y_2$. We again arrive to

$$|S| \leq n^2 - \Lambda(T_2) - \Lambda(A) - \frac{1}{2}\Lambda(P_2) \leq n^2 + \frac{c^2}{8p^2m} - \|A\|.$$

Suppose now that $c \leq n$. If $-\frac{c}{m} \leq n$, then $|S| \leq \frac{1}{2}n^2$ by excluding A alone. So $-\frac{c}{m} > n$. Then $\|A\| = \frac{n(2n-mn-2c)}{2}$ and we get, using $c \leq n$ and $x + y \geq 2\sqrt{xy}$ for $x, y > 0$,

$$|S| \leq n^2 + \frac{c^2}{8p^2m} - \|A\| = \frac{c^2}{8p^2m} + \frac{n^2m}{2} + cn \leq cn - \frac{cn}{2p} \leq \left(1 - \frac{1}{2p}\right)n^2.$$

We may then assume $c > n$. The case $-\frac{c}{m} \leq n$ can be handled as the above $c \leq n$ and $-\frac{c}{m} \geq n$ case. Thus, we can assume $-\frac{c}{m} > n$. Then A is a triangle with $\|A\| = -\frac{(n-mn-c)^2}{2m}$ and

$$|S| \leq n^2 + \frac{c^2}{8p^2m} + \frac{(n-mn-c)^2}{2m} = \left(\frac{1}{8p^2} + \frac{1}{2}\right)\frac{c^2}{m} + \frac{n(m-1)}{m}c + \frac{n^2m}{2} + \frac{n^2}{2m}.$$

The quadratic function of c above is maximized when $c = -\frac{(m-1)n}{1+\frac{1}{4p^2}}$. Thus

$$|S| \le n^2 \left[\frac{4p^2}{4p^2+1} + \left(\frac{1}{2} - \frac{2p^2}{4p^2+1} \right) \left(m + \frac{1}{m} \right) \right] \le \left(1 - \frac{2}{4p^2+1} \right) n^2,$$

where the maximum is achieved when we choose $m = -1$ and thus $c = \frac{8p^2}{4p^2+1} n$.

This completes the proof. $\square$

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