

# The number of maximal sum-free subsets of integers

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## Abstract

Cameron and Erdős [6] raised the question of how many *maximal* sum-free sets there are in  $\{1, \dots, n\}$ , giving a lower bound of  $2^{\lfloor n/4 \rfloor}$ . In this paper we prove that there are in fact at most  $2^{(1/4+o(1))n}$  maximal sum-free sets in  $\{1, \dots, n\}$ . Our proof makes use of container and removal lemmas of Green [8, 9] as well as a result of Deshouillers, Freiman, Sós and Temkin [7] on the structure of sum-free sets.

## 1 Introduction

A fundamental notion in combinatorial number theory is that of a sum-free set: A set  $S$  of integers is *sum-free* if  $x + y \notin S$  for every  $x, y \in S$  (note  $x$  and  $y$  are not necessarily distinct here). The topic of sum-free sets of integers has a long history. Indeed, in 1916 Schur [19] proved that, if  $n$  is sufficiently large, then any  $r$ -colouring of  $[n] := \{1, \dots, n\}$  yields a monochromatic triple  $x, y, z$  such that  $x + y = z$ .

Note that both the set of odd numbers in  $[n]$  and the set  $\{\lfloor n/2 \rfloor + 1, \dots, n\}$  are maximal sum-free sets. (A sum-free subset of  $[n]$  is *maximal* if it is not properly contained in another sum-free subset of  $[n]$ .) By considering all possible subsets of one of these maximal sum-free sets, we see that  $[n]$  contains at least  $2^{\lfloor n/2 \rfloor}$  sum-free sets. Cameron and Erdős [5] conjectured that in fact  $[n]$  contains only  $O(2^{n/2})$  sum-free sets. The conjecture was proven independently by Green [8] and Sapozhenko [16]. Recently, a refinement of the Cameron–Erdős conjecture was proven in [1], giving an upper bound on the number of sum-free sets in  $[n]$  of size  $m$  (for each  $1 \leq m \leq \lfloor n/2 \rfloor$ ).

Let  $f(n)$  denote the number of sum-free subsets of  $[n]$  and  $f_{\max}(n)$  denote the number of maximal sum-free subsets of  $[n]$ . Recall that the sum-free subsets of  $[n]$  described above lie in

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just two maximal sum-free sets. This led Cameron and Erdős [6] to ask whether the number of maximal sum-free subsets of  $[n]$  is “substantially smaller” than the total number of sum-free sets. In particular, they asked whether  $f_{\max}(n) = o(f(n))$  or even  $f_{\max}(n) \leq f(n)/2^{\varepsilon n}$  for some constant  $\varepsilon > 0$ . Łuczak and Schoen [14] answered this question, showing that  $f_{\max}(n) \leq 2^{n/2-2^{-28}n}$  for sufficiently large  $n$ . More recently, Wolfowitz [20] proved that  $f_{\max}(n) \leq 2^{3n/8+o(n)}$ .

In the other direction, Cameron and Erdős [6] observed that  $f_{\max}(n) \geq 2^{\lfloor n/4 \rfloor}$ . Indeed, let  $m = n$  or  $m = n - 1$ , whichever is even. Let  $S$  consist of  $m$  together with precisely one number from each pair  $\{x, m - x\}$  for odd  $x < m/2$ . Then  $S$  is sum-free. Moreover, although  $S$  may not be maximal, no further odd numbers less than  $m$  can be added, so distinct  $S$  lie in distinct maximal sum-free subsets of  $[n]$ .

We prove that this lower bound is in fact, ‘asymptotically’, the correct bound on  $f_{\max}(n)$ .

**Theorem 1.1.** *There are at most  $2^{(1/4+o(1))n}$  maximal sum-free sets in  $[n]$ . That is,*

$$f_{\max}(n) = 2^{(1/4+o(1))n}.$$

The proof of Theorem 1.1 makes use of ‘container’ and ‘removal’ lemmas of Green [8, 9] as well as a result of Deshouillers, Freiman, Sós and Temkin [7] on the structure of sum-free sets (see Section 2 for an overview of the proof).

Next we provide another collection of maximal sum-free sets in  $[n]$ . Suppose that  $4|n$  and set  $I_1 := \{n/2 + 1, \dots, 3n/4\}$  and  $I_2 := \{3n/4 + 1, \dots, n\}$ . First choose the element  $n/4$  and a set  $S \subseteq I_2$ . Then for every  $x \in I_2 \setminus S$ , choose  $x - n/4 \in I_1$ . The resulting set is sum-free but may not be maximal. However, no further element in  $I_2$  can be added, thus distinct  $S$  lie in distinct maximal sum-free sets in  $[n]$ . There are  $2^{|I_2|} = 2^{n/4}$  ways to choose  $S$ .

It would be of interest to establish whether  $f_{\max}(n) = O(2^{n/4})$ .

**Question 1.2.** Does  $f_{\max}(n) = O(2^{n/4})$ ?

In a forthcoming paper [2] we consider the analogous problem for maximal sum-free sets in abelian groups.

**Notation:** Given a set  $A \subseteq [n]$ , denote by  $f_{\max}(A)$  the number of maximal sum-free subsets of  $[n]$  that lie in  $A$  and by  $\min(A)$  the minimum element of  $A$ . Let  $1 \leq p < q \leq n$  be integers, denote  $[p, q] := \{p, p + 1, \dots, q\}$ . Denote by  $E$  the set of all even numbers in  $[n]$  and by  $O$  the set of all odd numbers in  $[n]$ . A triple  $x, y, z \in [n]$  is called a *Schur triple* if  $x + y = z$  (here  $x = y$  is allowed).

Throughout, all graphs considered are simple unless stated otherwise. We say that a graph  $G$  is a *graph possibly with loops* if  $G$  can be obtained from a simple graph by adding at most one loop at each vertex. Given a vertex  $x$  in  $G$ , we write  $\deg_G(x)$  for the *degree of  $x$  in  $G$* . Note that a loop at  $x$  contributes two to the degree of  $x$ . We write  $\delta(G)$  for the *minimum degree of  $G$*  and  $\Delta(G)$  for the *maximum degree of  $G$* . Given a graph  $G$ , denote by  $\text{MIS}(G)$  the number of maximal independent sets in  $G$ . Given  $T \subseteq V(G)$ , denote by  $\Gamma(T)$  the external neighbourhood of  $T$ , i.e.  $\Gamma(T) := \{v \in V(G) \setminus T : \exists u \in T, uv \in E(G)\}$ . Denote by  $G[T]$  the induced subgraph of  $G$  on the vertex set  $T$  and let  $G \setminus T$  denote the induced subgraph of  $G$  on the vertex set  $V(G) \setminus T$ . Denote by  $E(T)$  the set of edges in  $G$  spanned by  $T$  and by  $E(T, V(G) \setminus T)$  the set of edges in  $G$  with exactly one vertex in  $T$ .

## 2 Overview of the proof and preliminary results

### 2.1 Proof overview

We prove Theorem 1.1 in Section 3. A key tool in the proof is the following container lemma of Green [8] for sum-free sets. The first container-type result in the area (for counting sum-free subsets of  $\mathbb{Z}_p$ ) was given by Green and Ruzsa [10].

**Lemma 2.1** (Proposition 6 in [8]). *There exists a family  $\mathcal{F}$  of subsets of  $[n]$  with the following properties.*

- (i) *Every member of  $\mathcal{F}$  has at most  $o(n^2)$  Schur triples.*
- (ii) *If  $S \subseteq [n]$  is sum-free, then  $S$  is contained in some member of  $\mathcal{F}$ .*
- (iii)  *$|\mathcal{F}| = 2^{o(n)}$ .*
- (iv) *Every member of  $\mathcal{F}$  has size at most  $(1/2 + o(1))n$ .*

We refer to the elements of  $\mathcal{F}$  from Lemma 2.1 as *containers*. In [8], condition (iv) was not stated explicitly. However, it follows immediately from (i) by, for example, applying Theorem 2.2 and Lemma 2.3 below. Lemma 2.1 can also be derived from a general theorem of Balogh, Morris and Samotij [3], and independently Saxton and Thomason [18] with better bounds in (i) and (iii).

Note that conditions (ii) and (iii) in Lemma 2.1 imply that, to prove Theorem 1.1, it suffices to show that every member of  $\mathcal{F}$  contains at most  $2^{n/4+o(n)}$  maximal sum-free subsets of  $[n]$ . For this purpose, we need to get a handle on the structure of the containers; this is made precise in Lemma 2.4 below. The following theorem of Deshouillers, Freiman, Sós and Temkin [7] provides a structural characterisation of the sum-free sets in  $[n]$ .

**Theorem 2.2.** *Every sum-free set  $S$  in  $[n]$  satisfies at least one of the following conditions:*

- (i)  $|S| \leq 2n/5 + 1$ ;
- (ii)  $S$  consists of odd numbers;
- (iii)  $|S| \leq \min(S)$ .

We also need the following removal lemma of Green [9] for sum-free sets. (A simpler proof of Lemma 2.3 was later given by Král', Serra and Vena [13].)

**Lemma 2.3** (Corollary 1.6 in [9]). *Suppose that  $A \subseteq [n]$  is a set containing  $o(n^2)$  Schur triples. Then, there exist  $B$  and  $C$  such that  $A = B \cup C$  where  $B$  is sum-free and  $|C| = o(n)$ .*

Together, Theorem 2.2 and Lemma 2.3 yield the following structural result on containers of size close to  $n/2$ .

**Lemma 2.4.** *If  $A \subseteq [n]$  has  $o(n^2)$  Schur triples and  $|A| = (\frac{1}{2} - \gamma)n$  with  $\gamma = \gamma(n) \leq 1/11$ , then one of the following conditions holds.*

- (a) *All but  $o(n)$  elements of  $A$  are contained in the interval  $[(1/2 - \gamma)n, n]$ .*
- (b) *Almost all elements of  $A$  are odd, i.e.  $|A \setminus O| = o(n)$ .*

*Proof.* Apply Lemma 2.3 to  $A$ ; we have  $A = B \cup C$  with  $B$  sum-free and  $|C| = o(n)$ . Apply Theorem 2.2 to  $B$ . Alternative (i) is impossible, since  $|B| \geq (1 - o(1))|A| > 2n/5 + 1$ . If alternative (ii) occurs, then we have  $|A \setminus O| \leq |C| = o(n)$ . If alternative (iii) occurs, then  $\min(B) \geq |B| \geq (1/2 - \gamma - o(1))n$ . So all but except  $o(n)$  elements of  $A$  are contained in  $[(1/2 - \gamma)n, n]$ .  $\square$

We remark that Lemma 2.4 was already essentially proven in [8] (without applying Lemma 2.3). Note that  $\gamma$  could be negative in Lemma 2.4. The upper bound  $1/11$  on  $\gamma$  here can be relaxed to any constant smaller than  $1/10$  (but not to a constant bigger than  $1/10$ ). Roughly speaking, Lemma 2.4 implies that every container  $A \in \mathcal{F}$  is such that (a) most elements of  $A$  lie in  $[n/2, n]$ , (b) most elements of  $A$  are odd or (c)  $|A|$  is significantly smaller than  $n/2$ . Thus, the proof of Theorem 1.1 splits into three cases depending on the structure of our container. In each case, we give an upper bound on the number of maximal sum-free sets in a container by counting the number of maximal independent sets in various auxiliary graphs. (Similar techniques were used in [20], and in the graph setting in [4].) In the following subsection we collect together a number of results that are useful for this.

## 2.2 Maximal independent sets in graphs

Moon and Moser [15] showed that for any graph  $G$ ,  $\text{MIS}(G) \leq 3^{|G|/3}$ . We will need a looped version of this statement. Since any vertex with a loop cannot be in an independent set, the following statement is an immediate consequence of Moon and Moser's result.

**Proposition 2.5.** *Let  $G$  be a graph possibly with loops. Then*

$$\text{MIS}(G) \leq 3^{|G|/3}.$$

When a graph is triangle-free, the bound in Proposition 2.5 can be improved significantly. A result of Hujter and Tuza [11] states that for any triangle-free graph  $G$ ,

$$\text{MIS}(G) \leq 2^{|G|/2}. \tag{1}$$

The following lemma is a slight modification of this result for graphs with 'few' triangles.

**Lemma 2.6.** *Let  $G$  be a graph possibly with loops. If there exists a set  $T$  such that  $G \setminus T$  is triangle-free, then*

$$\text{MIS}(G) \leq 2^{|G|/2 + |T|/2}.$$

*Proof.* Every maximal independent set in  $G$  can be obtained in the following two steps:

- (1) Choose an independent set  $S \subseteq T$ .
- (2) Extend  $S$  in  $V(G) \setminus T$ , i.e. choose a set  $R \subseteq V(G) \setminus T$  such that  $R \cup S$  is a maximal independent set in  $G$ .

Note that although every maximal independent set in  $G$  can be obtained in this way, it is not necessarily the case that given an arbitrary independent set  $S \subseteq T$ , there exists a set  $R \subseteq V(G) \setminus T$  such that  $R \cup S$  is a maximal independent set in  $G$ . Notice that if  $R \cup S$  is

maximal,  $R$  is also a maximal independent set in  $G \setminus \{T \cup \Gamma(S)\}$ . The number of choices for  $S$  in (1) is at most  $2^{|T|}$ . Since  $G \setminus \{T \cup \Gamma(S)\}$  is triangle-free, by the Hujter–Tuza bound, the number of extensions in (2) is at most  $2^{(|G|-|T|)/2}$ . Thus, we have  $\text{MIS}(G) \leq 2^{|T|} \cdot 2^{(|G|-|T|)/2} = 2^{|G|/2+|T|/2}$ .  $\square$

The following lemma gives an improvement on Proposition 2.5 for graphs that are ‘too sparse and almost regular’. The proof uses an elegant and simple idea of Sapozhenko [17], see [12] for a closely-related result.

**Lemma 2.7.** *Let  $k \geq 1$  and let  $G$  be a graph on  $n$  vertices possibly with loops. Suppose that  $\Delta(G) \leq k\delta(G)$  where  $\delta(G) \geq f(n)$  for some real valued function  $f$  with  $f(n) \rightarrow \infty$  as  $n \rightarrow \infty$ . Then*

$$\text{MIS}(G) \leq 3^{\binom{k}{k+1} \frac{n}{3} + o(n)}.$$

*Proof.* Fix a maximal independent set  $I$  in  $G$  and set  $b := \delta(G)^{1/2}$ . We will repeat the following process as many times as possible. Let  $V_1 := V(G)$ . At the  $i$ -th step, for  $i \geq 1$ , choose  $v_i \in V_i \cap I$  such that  $\deg_{G[V_i]}(v_i) \geq b$  and set  $V_{i+1} := V_i \setminus (\{v_i\} \cup \Gamma(v_i))$ . This process is repeated  $j \leq n/b$  times. Let  $U := V_{j+1}$  be the resulting set. Define  $Z := \{v \in U : \deg_{G[U]}(v) < b\}$ . Notice that  $\deg_{G[U]}(v) < b$  for all  $v \in I \cap U$ , hence  $I \cap U \subseteq Z$ . We have

$$\delta(G) \cdot |Z| \leq \sum_{v \in Z} \deg(v) = 2|E(Z)| + |E(Z, V \setminus Z)| \leq b|Z| + \Delta(G) \cdot (n - |Z|).$$

Hence,

$$|Z| \leq \frac{\Delta(G) \cdot n}{\delta(G) + \Delta(G) - b} \leq \frac{k}{k+1}n + \frac{2n}{b}. \quad (2)$$

By construction of  $U$ , no vertex in  $I \setminus U$  has a neighbour in  $U$ . So as  $Z \subseteq U$ , no vertex in  $Z$  is adjacent to  $I \setminus U$ . Together with the fact that  $I$  is maximal, this implies that  $I \cap U$  is a maximal independent set in  $G[Z]$ . By the above process, every maximal independent set  $I$  in  $G$  is determined by a set  $I \setminus U$  of at most  $n/b$  vertices and a maximal independent set in  $G[Z]$ . Note that  $n/b = o(n)$ . Thus, Proposition 2.5 and (2) imply that

$$\text{MIS}(G) \leq \sum_{0 \leq i \leq n/b} \binom{n}{i} 3^{\binom{k}{k+1} \frac{n}{3} + \frac{2n}{3b}} \leq 3^{\binom{k}{k+1} \frac{n}{3} + o(n)}. \quad (3)$$

$\square$

Note that one could relax the minimum degree condition in Lemma 2.7 to (for example) a large constant, at the expense of a worse upper bound on  $\text{MIS}(G)$ . However, Lemma 2.7 in its current form suffices for our applications.

### 3 Proof of Theorem 1.1

Let  $\mathcal{F}$  be the family of containers obtained from Lemma 2.1. Recall that given a set  $A \subseteq [n]$ ,  $f_{\max}(A)$  denotes the number of maximal sum-free subsets of  $[n]$  that lie in  $A$ . Since every sum-free subset of  $[n]$  is contained in some member of  $\mathcal{F}$  and  $|\mathcal{F}| = 2^{o(n)}$ , it suffices to show that  $f_{\max}(A) \leq 2^{(1/4+o(1))n}$  for every container  $A \in \mathcal{F}$ .

Lemmas 2.1 and 2.4 imply that every container  $A \in \mathcal{F}$  satisfies at least one of the following conditions:

- (a)  $|A| \leq (1/2 - 1/11)n \leq 0.45n$   
or one of the following holds for some  $-o(1) \leq \gamma = \gamma(n) \leq 1/11$ :
- (b)  $|A| = (\frac{1}{2} - \gamma)n$  and  $|A \cap [(1/2 - \gamma)n]| = o(n)$ ;
- (c)  $|A| = (\frac{1}{2} - \gamma)n$  and  $|A \setminus O| = o(n)$ .

We deal with each of the three cases separately.

For any subsets  $B, S \subseteq [n]$ , let  $L_S[B]$  be the *link graph of  $S$  on  $B$*  defined as follows. The vertex set of  $L_S[B]$  is  $B$ . The edge set of  $L_S[B]$  consists of the following two types of edges:

- (i) Two vertices  $x$  and  $y$  are adjacent if there exists an element  $z \in S$  such that  $\{x, y, z\}$  forms a Schur triple;
- (ii) There is a loop at a vertex  $x$  if  $\{x, x, z\}$  forms a Schur triple for some  $z \in S$  or if  $\{x, z, z'\}$  forms a Schur triple for some  $z, z' \in S$ .

The following simple result will be applied in all three cases of our proof.

**Lemma 3.1.** *Suppose that  $B, S$  are both sum-free subsets of  $[n]$ . If  $I \subseteq B$  is such that  $S \cup I$  is a maximal sum-free subset of  $[n]$ , then  $I$  is a maximal independent set in  $G := L_S[B]$ .*

*Proof.* First notice that  $I$  is an independent set in  $G$ , since otherwise  $S \cup I$  is not sum-free. Suppose to the contrary that there exists a vertex  $v \notin I$  such that  $I' := I \cup \{v\}$  is still an independent set in  $G$ . Then since  $I' \subseteq B$  is sum-free, the definition of  $G$  implies that  $S \cup I'$  is a sum-free set in  $[n]$  containing  $S \cup I$ , a contradiction to the maximality of  $S \cup I$ .  $\square$

#### 3.1 Small containers

The following lemma deals with containers of ‘small’ size.

**Lemma 3.2.** *If  $A \in \mathcal{F}$  has size at most  $0.45n$ , then  $f_{\max}(A) = o(2^{n/4})$ .*

*Proof.* Lemma 2.1 (i) implies that we can apply Lemma 2.3 to  $A$  to obtain that  $A = B \cup C$  where  $B$  is sum-free and  $|C| = o(n)$ . Notice crucially that every maximal sum-free subset of  $[n]$  in  $A$  can be built in the following two steps:

- (1) Choose a sum-free set  $S$  in  $C$ ;
- (2) Extend  $S$  in  $B$  to a maximal one.

(As in Lemma 2.6, note that it is not necessarily the case that given an arbitrary sum-free set  $S \subseteq C$ , there exists a set  $R \subseteq B$  such that  $R \cup S$  is a maximal sum-free set in  $[n]$ .)

The number of choices for  $S$  is at most  $2^{|C|} = 2^{o(n)}$ . For a fixed  $S$ , denote by  $N(S, B)$  the number of extensions of  $S$  in  $B$  in Step (2). It suffices to show that for any given sum-free set

$S \subseteq C$ ,  $N(S, B) \leq 2^{0.249n}$ . Let  $G := L_S[B]$  be the link graph of  $S$  on  $B$ . Since  $|A| \leq 0.45n$  and  $S$  and  $B$  are sum-free, Lemma 3.1 and Proposition 2.5 imply that

$$N(S, B) \leq \text{MIS}(G) \leq 3^{|B|/3} \leq 3^{|A|/3} \leq 3^{0.45n/3} \ll 2^{0.249n}.$$

□

## 3.2 Large containers

We now turn our attention to containers of relatively large size.

**Lemma 3.3.** *Let  $-o(1) \leq \gamma = \gamma(n) \leq 1/11$ . If  $A \subseteq [n]$  has  $o(n^2)$  Schur triples,  $|A| = (\frac{1}{2} - \gamma)n$  and  $|A \cap [(1/2 - \gamma)n]| = o(n)$ , then*

$$f_{\max}(A) \leq 2^{(1/4+o(1))n}.$$

*Proof.* Let  $A \in \mathcal{F}$  be as in the statement of the lemma. Let  $A_1 := A \cap [\lfloor n/2 \rfloor]$  and  $A_2 := A \setminus A_1$ . Since  $|A \cap [(1/2 - \gamma)n]| = o(n)$ , we have that  $|A_1| \leq (\gamma + o(1))n$ . Every maximal sum-free subset of  $[n]$  in  $A$  can be built from choosing a sum-free set  $S \subseteq A_1$  and extending  $S$  in  $A_2$ . The number of choices for  $S$  is at most  $2^{|A_1|}$ .

Let  $G := L_S[A_2]$  be the link graph of  $S$  on vertex set  $A_2$ . Since  $S$  and  $A_2$  are sum-free, Lemma 3.1 implies that  $N(S, A_2) \leq \text{MIS}(G)$ . Notice that  $G$  is triangle-free. Indeed, suppose to the contrary that  $z > y > x > n/2$  form a triangle in  $G$ . Then there exists  $a, b, c \in S$  such that  $z - y = a$ ,  $y - x = b$  and  $z - x = c$ , which implies  $a + b = c$  with  $a, b, c \in S$ . This is a contradiction to  $S$  being sum-free. Thus by (1) we have  $N(S, A_2) \leq \text{MIS}(G) \leq 2^{|A_2|/2}$ . Then we have

$$f_{\max}(A) \leq 2^{|A_1|+|A_2|/2} = 2^{|A_1|+(1/2-\gamma)n-|A_1|/2} = 2^{n/4+(|A_1|-\gamma n)/2} \leq 2^{n/4+o(n)},$$

where the last inequality follows since  $|A_1| \leq (\gamma + o(1))n$ . □

**Lemma 3.4.** *If  $A \in \mathcal{F}$  such that  $|A \setminus O| = o(n)$ , then*

$$f_{\max}(A) \leq 2^{(1/4+o(1))n}.$$

*Proof.* Let  $A \in \mathcal{F}$  be as in the statement of the lemma. Notice that if  $S \subseteq T \subseteq [n]$  then  $f_{\max}(S) \leq f_{\max}(T)$ . Using this fact, we may assume that  $A = O \cup C$  with  $C \subseteq E$  and  $|C| = o(n)$ . Similarly to before, every maximal sum-free subset of  $[n]$  in  $A$  can be built from choosing a sum-free set  $S \subseteq C$  (at most  $2^{|C|} = 2^{o(n)}$  choices) and extending  $S$  in  $O$  to a maximal one. Fix an arbitrary sum-free set  $S$  in  $C$  and let  $G := L_S[O]$  be the link graph of  $S$  on vertex set  $O$ . Since  $O$  is sum-free, by Lemma 3.1 we have that  $N(S, O) \leq \text{MIS}(G)$ . It suffices to show that  $\text{MIS}(G) \leq 2^{n/4+o(n)}$ . We will achieve this in two cases depending on the size of  $S$ .

**Case 1:**  $|S| \geq n^{1/4}$ .

In this case, we will show that  $G$  is ‘not too sparse and almost regular’. Then we apply Lemma 2.7.

We first show that  $\delta(G) \geq |S|/2$  and  $\Delta(G) \leq 2|S| + 2$ , thus  $\Delta(G) \leq 6\delta(G)$ . Let  $x$  be any vertex in  $O$ . If  $s \in S$  such that  $s < \max\{x, n - x\}$  then at least one of  $x - s$  and  $x + s$  is adjacent to  $x$  in  $G$ . If  $s \in S$  such that  $s \geq \max\{x, n - x\}$  then  $s - x$  is adjacent to  $x$  in  $G$ . By considering all  $s \in S$  this implies that  $\deg_G(x) \geq |S|/2$  (we divide by 2 here as an edge  $xy$  may arise from two different elements of  $S$ ). For the upper bound consider  $x \in O$ . If  $xy \in E(G)$  then  $y = x + s, x - s$  or  $s - x$  for some  $s \in S$  and only two of these terms are positive. Further, there may be a loop at  $x$  in  $G$  (contributing 2 to the degree of  $x$  in  $G$ ). Thus,  $\deg_G(x) \leq 2|S| + 2$ , as desired.

Since  $\delta(G) \geq |S|/2 \geq n^{1/4}/2$  we can apply Lemma 2.7 to  $G$  with  $k = 6$ . Hence,

$$\text{MIS}(G) \leq 3^{\left(\frac{6}{7}\right)\frac{n/2}{3} + o(n)} \ll 2^{0.24n + o(n)} = o(2^{n/4}).$$

**Case 2:**  $|S| \leq n^{1/4}$ .

In this case, it suffices to show that  $G$  has very few,  $o(n)$ , triangles, since then by applying Lemma 2.6 with  $T$  being the vertex set of all triangles in  $G$ , we have  $|T| = o(n)$  and then  $\text{MIS}(G) \leq 2^{n/4 + o(n)}$ . Recall that for each edge  $xy$  in  $G$ , at least one of the evens  $x + y$  and  $|x - y|$  is in  $S$ . We call  $xy$  a BLUE edge if  $|x - y|$  is in  $S$  and a RED edge if  $|x - y| \notin S$  and  $x + y \in S$ .

**Claim 3.5.** *Each triangle in  $G$  contains either 0 or 2 BLUE edges.*

*Proof.* Let  $xyz$  be a triangle in  $G$  with  $x < y < z$ . Suppose that  $xyz$  has only one BLUE edge  $xz$ . Then  $s_1 := z - x, s_2 := x + y$  and  $s_3 := y + z$  are elements of  $S$  and  $s_1 + s_2 = s_3$ , a contradiction to  $S$  being sum-free. All other cases, including when all the edges are BLUE, are similar, we omit the proof here.  $\square$

Consider an arbitrary triple  $\{s_1, s_2, s_3\}$  in  $S$  (where  $s_1, s_2$  and  $s_3$  are not necessarily distinct). We say that  $\{s_1, s_2, s_3\}$  forces a triangle  $\mathcal{T}$  in  $G$  if the vertex set  $\{x, y, z\}$  of  $\mathcal{T}$  is such that  $s_1, x, y; s_2, y, z$  and;  $s_3, x, z$  form Schur triples. Note that by definition of  $G$ , every triangle in  $G$  is forced by some triple in  $S$ .

Fix an arbitrary triple  $\{s_1, s_2, s_3\}$  in  $S$ . We will show that  $\{s_1, s_2, s_3\}$  forces at most 24 triangles in  $G$ . This then implies that  $G$  has at most  $24|S|^3 = o(n)$  triangles as desired.

By Claim 3.5, a triangle  $xyz$  with  $x < y < z$  can only be one of the following four types: (1) all edges are RED; (2)  $xy$  is the only RED edge; (3)  $yz$  is the only RED edge; (4)  $xz$  is the only RED edge.

It suffices to show that  $\{s_1, s_2, s_3\}$  can force at most 6 triangles of each type. We show it only for Type (1), the other types are similar. Suppose that  $xyz$  is a Type (1) triangle

forced by  $\{s_1, s_2, s_3\}$ . Set  $M := \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ . Then  $\mathbf{u} = (x, y, z)^T$  is a solution to  $M \cdot \mathbf{u} = \mathbf{s}$

for some  $\mathbf{s}$  whose entries are precisely the elements of  $\{s_1, s_2, s_3\}$ .

Since  $\det(M) = 2 \neq 0$ , if a solution  $\mathbf{u}$  exists to  $M \cdot \mathbf{u} = \mathbf{s}$ , it should be unique. The number of choices for  $\mathbf{s}$ , for fixed  $\{s_1, s_2, s_3\}$ , is  $3! = 6$ . Thus in total there are at most 6 triangles of Type (1) forced by  $\{s_1, s_2, s_3\}$ .

This completes the proof of Lemma 3.4.  $\square$

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