Rainbow spanning trees in properly coloured complete graphs

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Abstract

In this short note, we study pairwise edge-disjoint rainbow spanning trees in properly edge-coloured complete graphs, where a graph is rainbow if its edges have distinct colours. Brualdi and Hollingsworth conjectured that every K_n properly edge-coloured by $n-1$ colours has $n/2$ edge-disjoint rainbow spanning trees. Kaneko, Kano and Suzuki later suggested this should hold for every properly edge-coloured K_n. Improving the previous best known bound, we show that every properly edge-coloured K_n contains $\Omega(n)$ pairwise edge-disjoint rainbow spanning trees.

Independently, Pokrovskiy and Sudakov recently proved that every properly edge-coloured K_n contains $\Omega(n)$ isomorphic pairwise edge-disjoint rainbow spanning trees.

1 Introduction

Given a properly edge-coloured K_n, a *rainbow spanning tree* is a tree with vertex set $V(K_n)$ whose edges have distinct colours. Each properly edge-coloured K_n clearly contains many rainbow spanning trees – for example, any $(n-1)$-edge star in K_n is such a tree. How many *edge-disjoint* rainbow spanning trees can we find in any properly edge-coloured K_n? Brualdi and Hollingsworth [2] conjectured that every K_n properly edge-coloured by $n-1$ colours contains $n/2$ edge-disjoint rainbow spanning trees (see also Constantine [4]). Note that, in such a colouring, each colour appears on exactly $n/2$ edges to form a monochromatic perfect matching. Brualdi and Hollingsworth [2] showed that there are at least two edge-disjoint rainbow spanning trees in any K_n properly edge-coloured by $n-1$ colours.

Kaneko, Kano and Suzuki [6] expanded the Brualdi-Hollingsworth conjecture, suggesting that any properly edge-coloured K_n (using any number of colours) should contain $n/2$ edge-disjoint rainbow spanning trees. In such graphs, Kaneko, Kano and Suzuki [6] showed that there are at least three edge-disjoint rainbow spanning trees. Akbari and Alipour [1] studied edge-disjoint rainbow spanning trees using weaker conditions still, showing that any edge-coloured K_n, where each colour is only constrained to appear at most $n/2$ times, contains at least two edge-disjoint rainbow spanning trees.

Recent progress has seen far more edge-disjoint rainbow spanning trees found in edge-coloured complete graphs. For some small $\varepsilon > 0$, Horn [5] proved that every properly $(n-1)$-edge-coloured K_n contains at least εn edge-disjoint rainbow spanning trees. Carraher, Hartke and Horn [3]

showed that every edge-coloured K_n where each colour appears at most $n/2$ times contains at least $\lfloor n/(1000\log n)\rfloor$ edge-disjoint rainbow spanning trees. Here, we consider the intermediate of these conditions, and show that any properly edge-coloured K_n contains linearly many edge-disjoint rainbow spanning trees.

Theorem 1.1. *Every properly edge-coloured K_n contains at least $n/10^{12}$ pairwise edge-disjoint rainbow spanning trees.*

We note that our methods are much shorter than those previously capable of finding many edge-disjoint rainbow spanning trees. In particular, Section 4 alone shows that any properly $(n-1)$-edge-coloured K_n contains linearly many edge-disjoint rainbow spanning trees.

Essentially, to prove Theorem 1.1, we iteratively remove rainbow spanning trees from K_n while ensuring that the remaining minimum degree does not decrease too much and that we do not use up too many colours. We have two cases, depending on the colouring of K_n. We show (in Lemma 2.5) that every properly edge-coloured K_n either contains many colours which appear on linearly many edges or its colours can be grouped into classes which play a similar role. We use different embedding strategies for these cases (in Lemmas 2.6 and 2.7).

While finishing our work, we discovered that, using different methods, Pokrovskiy and Sudakov [7] recently proved that every properly edge-coloured K_n contains at least $n/10^6$ edge-disjoint rainbow copies of a certain spanning tree with radius 2.

Organisation: The rest of the paper is organised as follows. In Section 2, we present our main lemmas and show that they imply Theorem 1.1. These lemmas, Lemmas 2.5, 2.6 and 2.7, are proved in Sections 3, 4 and 5 respectively.

2 Preliminaries

Notation. For a graph G, denote by $|G|$ the number of vertices in G and by $\nu(G)$ the size of a largest matching in G. For each $A \subseteq V(G)$, let $G[A]$ and $G - A$ be the induced subgraph of G on the vertex set A and $V(G) \setminus A$ respectively. When F is a spanning subgraph of G, denote by $G \setminus F$ the spanning subgraph of G with edge set $E(G) \setminus E(F)$. For any set of colours U from an edge-coloured graph G, let G_U be the maximal subgraph of G whose edges have colour in U. We omit floor and ceiling signs when they are not essential.

The main tool we use to find rainbow spanning trees is the following result of Schrijver [8] and Suzuki [9].

Theorem 2.1. *Let G be an n-vertex edge-coloured graph. If, for every $1 \le s \le n$ and for every partition S of $V(G)$ into s parts, there are at least $s - 1$ edges of different colours between the parts of S, then G contains a rainbow spanning tree.* □

Roughly speaking, we will iteratively find and remove edge-disjoint spanning trees in an edge-coloured K_n. By applying Theorem 2.1 to a large subgraph on vertices which have large remaining degree, and using a matching to extend the resulting tree to a spanning tree of K_n, we can find spanning rainbow trees with bounded maximum degree in which vertices with small degree in the remaining subgraph of K_n appear as leaves in each new rainbow spanning tree. This allows the iterative removal of spanning trees without decreasing the minimum degree too drastically. This iteration is carried out for the key lemma, Lemma 4.1.

The embedding differs depending on the colouring of K_n. To describe our cases, we use the following definitions.

Definition 2.2. *Let $\alpha > 0$. We say a class of colours U is α-rich in an edge-coloured graph G if $\nu(G_U) \ge \alpha|G|$.*

Definition 2.3. *Let $\alpha > 0$. We say an edge-coloured graph G is α-well-coloured if its colours can be partitioned into $|G| - 1$ α-rich classes.*

Definition 2.4. *Let $\alpha > 0$ and $t \in \mathbb{N}$. We say an edge-coloured graph G is (α, t)-robustly-coloured if, despite the removal of any t rainbow forests, G remains α-well-coloured.*

If a graph does not have many rich colours, then it is robustly coloured, as follows.

Lemma 2.5. *Let $0 < \alpha \leq 1/8$ and $\beta \leq \alpha/4$. Then, for any properly edge-coloured K_n, either*
  (i) there are at least $n - 16\beta n$ colours that are α-rich, or
  (ii) K_n is $(\beta, \beta n)$-robustly-coloured.

Let K_n be properly edge-coloured. Letting $\alpha = 1/8$ and $\beta = \alpha/2400$, apply Lemma 2.5. If *(ii)* in Lemma 2.5 holds for K_n, then the following lemma shows that K_n contains at least $n/10^{12}$ edge-disjoint rainbow spanning trees.

Lemma 2.6. *Let $0 < \beta \leq 1/2$. Then, every $(\beta, \beta n)$-robustly-coloured K_n contains at least $\beta^2 n/2500$ edge-disjoint rainbow spanning trees.*

If, however, *(i)* in Lemma 2.5 holds for K_n, then the following lemma shows that K_n contains at least $n/10^{12}$ edge-disjoint rainbow spanning trees.

Lemma 2.7. *Let $0 < \alpha \leq 1/2$. Then, every properly edge-coloured K_n with at least $n - \alpha n/150$ α-rich colours contains at least $\alpha^2 n/10^6$ edge-disjoint rainbow spanning trees.*

Thus, in either case, K_n contains at least $n/10^{12}$ edge-disjoint rainbow spanning trees. Therefore, to prove Theorem 1.1 it is sufficient to prove Lemmas 2.5, 2.6 and 2.7, which we do in the remaining three sections.

3 Partitioning the colours

Before proving Lemma 2.5, we show that any graph without a rich colour, but with many edges, has a rich class of colours, as follows.

Proposition 3.1. *Let $\alpha > 0$ and $n \in \mathbb{N}$. Any properly edge-coloured n-vertex graph G with at least $4\alpha n^2$ edges and no α-rich colour has an α-rich class U of colours with $|E(G_U)| \leq 4\alpha n$.*

Proof. Let U be a (possibly empty) class of colours which maximises $\nu(G_U)$ subject to $|E(G_U)| \leq 2\nu(G_U)$ and $\nu(G_U) \leq 2\alpha n$. Assume that $\nu(G_U) < \alpha n$, for otherwise U satisfies the lemma. Let M be a maximal matching in G_U, so that $|M| < \alpha n$. If, for some $k \geq 1$, a colour c has k edges coloured c in $G - V(M)$ and at most k edges with a vertex in $V(M)$, we could add c to U to contradict the maximality of U (noting that $k \leq \alpha n$). As there are at most $|V(M)|n$ edges with a vertex in M, there are thus at most $2|V(M)|n < 4\alpha n^2$ edges in total, a contradiction. $\square$

Proof of Lemma 2.5. Let $C_1, \ldots, C_t$ be the α-rich colours in K_n. Let $r = n - 1 - t$. Suppose $r \geq 16\beta n$, for otherwise we are done. Let $\ell = \beta n$ and let $F \subset K_n$ be the union of any ℓ rainbow forests. Let G be the subgraph of K_n formed by removal of the edges of F and of any edges with colour C_i for some i. Note that $|E(G)| \geq n(n-1)/2 - |E(F)| - t \cdot n/2 \geq rn/2 - \beta n^2$.

Iteratively, remove as many disjoint β-rich classes $U_1, \ldots, U_s$ of colours from G as possible, subject to $|E(G_{U_i})| \leq \alpha n$, and call the resulting subgraph G'. Note that, as G has no α-rich colour, G' has no β-rich colour, C say, for otherwise $\{C\}$ would be β-rich with $|E(G'_{\{C\}})| \leq \alpha n$, a contradiction. If $s \geq r$, then, noting that each colour C_i has at least $\alpha n - \beta n \geq \beta n$ edges outside F, these classes along with C_i, $1 \leq i \leq t$, demonstrate that $K_n - F$ is β-well-coloured, and thus, as required, K_n is (β, ℓ)-robustly-coloured. Suppose then, that $s < r$.

As G' has no β-rich class of colours with at most $4\beta n \leq \alpha n$ edges, and no β-rich colour, by Proposition 3.1, $|E(G')| < 4\beta n^2$. On the other hand, as fewer than $r \cdot \alpha n$ edges were removed by the iterative process, $|E(G')| > rn/2 - \beta n^2 - \alpha rn \geq rn/4 \geq 4\beta n^2$, a contradiction. $\square$

4 Rainbow trees in robustly-coloured graphs

We will prove the following more general version of Lemma 2.6, as it will be useful later when proving Lemma 2.7.

Lemma 4.1. *Let $0 < \alpha \leq 1/2$, $n \in \mathbb{N}$ and $\ell \leq \alpha^2 n/2500$. Let G be an n-vertex properly edge-coloured graph. Let $G_i \subset G$, $1 \leq i \leq \ell$, be (α, ℓ)-robustly-coloured subgraphs of G satisfying $\delta(G_i) \geq (1 - \alpha/30)n$. Then, G contains edge-disjoint rainbow trees $T_1, \ldots, T_\ell$ so that, for each $1 \leq i \leq \ell$, T_i is a spanning tree of G_i.*

Note that Lemma 2.6 follows immediately from Lemma 4.1 by taking $\alpha = \beta$, $\ell = \beta^2 n/2500$ and $G = G_1 = \ldots = G_\ell = K_n$. Before proving Lemma 4.1, we first show that any well-coloured graph with large minimum degree contains a rainbow spanning tree.

Lemma 4.2. *Let $0 < \alpha \leq 1/6$, $n \geq 1/2\alpha^2$ and let G be a 2α-well-coloured n-vertex graph, with $\delta(G) \geq (1 - \alpha)n$. Then, G contains a rainbow spanning tree.*

Proof. Assume, for contradiction, that G has no rainbow spanning tree. By Theorem 2.1, for some $k \geq 2$, there is a partition $S = \{S_1, \ldots, S_k\}$ of $V(G)$ with $0 < |S_1| \leq |S_2| \leq \ldots \leq |S_k|$ which has at most $k - 2$ colours between the sets in S.

Consider a vertex $v \in S_1$. As $|S_1| \leq n/k \leq n/2$ and $\delta(G) \geq (1 - \alpha)n$, there are at least $n/2 - \alpha n \geq 2\alpha n$ edges from v to $\cup_{i=2}^k S_i$, all of which have different colours. Thus, $2\alpha n \leq k - 2$, so that $k \geq 2\alpha n$. In fact, then, $|S_1| \leq n/k \leq 1/2\alpha \leq \alpha n$. Therefore, v has at least $n - 2\alpha n$ neighbours in $\cup_{i=2}^k S_i$, and thus $k > (1 - 2\alpha)n$.

There must then be a set W of more than $(1 - 4\alpha)n$ singletons in S. Any 2α-rich colour class U has a matching with at least $2\alpha n$ edges, and so at least $4\alpha n$ vertices, one of which must be in W. Thus, there is some edge with colour in U across the partition S. As G has at least $n - 1$ disjoint such 2α-rich colour classes, there are at least $n - 1 > k - 2$ colours between the sets in S, a contradiction. $\square$

We now show that a careful application of Lemma 4.2 in a well-coloured graph can find a rainbow spanning tree with a maximum degree bound in which a previously chosen small set of vertices has particularly low degree.

Lemma 4.3. *Let $0 < \beta \leq \alpha \leq 1/15$ and $n \geq 1/\alpha^2$. Let G be a 6α-well-coloured n-vertex graph with $\delta(G) \geq 3\beta n$. Let $A \subseteq V(G)$ satisfy $|A| \leq \beta n$ and $\delta(G \setminus A) \geq (1 - \alpha)n$. Then, G contains a rainbow spanning tree with maximum degree at most $(1 - \alpha)n$ in which each vertex in A is a leaf.*

Proof. Choose $B \subset V(G)$ so that $A \subseteq B$ and $|B| = \alpha n$. Greedily, for each $v \in A$, pick a vertex $u_v \in V(G) \setminus A$ and a colour c_v so that $vu_v \in E(G)$ has colour c_v and the vertices and colours chosen are all distinct. This is possible as G is properly coloured with $\delta(G) \geq 3\beta n \geq 3|A|$. Then, for each $v \in B \setminus A$, greedily pick a vertex $u_v \in V(G) \setminus B$ and a colour c_v so that $vu_v \in E(G)$ has colour c_v and the vertices and colours u_v, c_v, $v \in B$, are all distinct. This is possible as G is properly coloured with $\delta(G \setminus A) \geq (1 - \alpha)n \geq 3|B|$.

Let H be the graph $G - B$ with any edges of colour c_v removed for each $v \in B$. Any matching has at most αn edges incident to B. Thus, any 6α-rich class U of colours in G is a 5α-rich class of colours in $G - B$. Furthermore, if such a class U contains no colour c_v, $v \in B$, then U is a 5α-rich class of colours in H. As the colours of G can be partitioned into $n - 1$ classes that are each 5α-rich, the colours of H have an (inherited) partition into $n - 1 - |B| = |H| - 1$ classes that are 5α-rich. Therefore, H is 5α-well-coloured.

Now, as each vertex in G has at most αn non-neighbours, and we removed edges of αn colours from $G - B$ to get H, $\delta(H) \geq |H| - 2\alpha n \geq (1 - 5\alpha/2)|H|$. Hence, by Lemma 4.2, H has a rainbow spanning tree, T say. Adding the edges $u_v v$, $v \in B$, to T, we get a rainbow spanning tree of G with maximum degree at most $(|T| - 1) + 1 \leq |H| = (1 - \alpha)n$ in which each vertex in A is a leaf, as desired. $\square$

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Proof of Lemma 4.1. Note the lemma is vacuous unless $\ell \geq 1$, so we may assume $n \geq 2500/\alpha^2$. Greedily, find edge-disjoint rainbow trees $T_1, \ldots, T_\ell$ under the rules that, for each $1 \leq i \leq \ell$,

- $\Delta(T_i) \leq (1 - \alpha/6)n$,

- every vertex in $F_i := \cup_{j<i} T_j$ with degree at least $\alpha n/12$ is a leaf in T_i, and

- T_i is a spanning tree of G_i.

This is possible, as follows. Let $1 \leq i \leq \ell$, and suppose we have found $T_1, \ldots, T_{i-1}$. Let $H_i := G_i - F_i$. From the rules above, each vertex in F_i has degree at most $\alpha n/12 + (1-\alpha/6)n + \ell \leq (1 - \alpha/15)n$. Thus, $\delta(H_i) \geq (1 - \alpha/30)n - (1 - \alpha/15)n = \alpha n/30$. Let $\beta = \alpha/90$, so that $\delta(H_i) \geq 3\beta n$.

As G_i is (α, ℓ)-robustly-coloured, H_i is α-well-coloured. Let $A_i \subseteq V(G_i)$ be the set of vertices in G_i with degree at least $\alpha n/12$ in F_i. Then, recalling that $\ell \leq \alpha^2 n/2500$,

$$|A_i| \leq 2\ell n/(\alpha n/12) \leq \alpha n/90 = \beta n \leq \delta(H_i)/3.$$

Furthermore, $\delta(H_i - A_i) \geq (1 - \alpha/30)n - |A| - \alpha n/12 \geq (1 - \alpha/6)n$. Applying Lemma 4.3 with $\alpha/6$, β, H_i and A_i playing the roles of α, β, G and A gives the required rainbow spanning tree T_i of H_i. □

5 Rainbow trees in graphs with many rich colours

Where K_n has $n - 1 - r$ many rich colours and r is small, we need to ensure that the iterative removal of rainbow spanning trees does not remove all the non-rich colours. We do this by first finding many edge-disjoint rainbow matchings of r edges using the non-rich colours, reserving one for each rainbow spanning tree we subsequently find. We find the matchings using the following proposition and lemma, before proving Lemma 2.7.

Proposition 5.1. *Let $r, n \in \mathbb{N}$. Any properly edge-coloured n-vertex graph G, with at least $rn/3$ edges, maximum degree at most $n/10$ and with no $(n/10)$-rich colour, contains a rainbow matching with r edges.*

Proof. Pick a maximal rainbow matching M in G. Say M has fewer than r edges, for otherwise we are done. Note that any edge with no vertex in $V(M)$ must have some colour from M. Therefore, G has at most $|E(M)| \cdot n/10 + |V(M)| \cdot n/10 < rn/3$ edges, a contradiction. □

Lemma 5.2. *Let $r, n \in \mathbb{N}$ satisfy $r \leq n/100$. Any properly edge-coloured n-vertex graph G with minimum degree at least r contains $n/100$ edge-disjoint rainbow matchings with r edges.*

Proof. Let $\ell = n/100$. Let $C_1, \ldots, C_s$ be maximally many $(1/20)$-rich colours in G subject to $s \leq r$, and let G' be the graph G with all edges of these colours removed. Let $v_1, \ldots, v_{s'}$ be maximally many vertices in G' with degree at least $n/20$ subject to $s + s' \leq r$. Let $G'' = G' - \{v_1, \ldots, v_s\}$ and $t = r - s - s' \geq 0$. As $\delta(G) \geq r$, $\delta(G'') \geq t$. Greedily, pick edge-disjoint rainbow matchings $M_1, \ldots, M_\ell$ with t edges in G''. This is possible, as, for each $0 \leq i < \ell$, $G'' - M_1 - \ldots - M_i$ has at least $t|G''|/2 - t \cdot \ell \geq t|G'|/3$ edges, and thus a suitable matching M_{i+1} exists by Proposition 5.1.

Now, greedily, for each M_i, add s' edges to M_i by adding, for each v_j, an edge xv_j in $E(G')$ with $x \notin V(M_i)$ such that M_i remains rainbow and the matchings $M_1, \ldots, M_\ell$ remain edge-disjoint. For each vertex v_j and matching M_i, when we seek to find x, at most ℓ neighbouring edges of v_j are in the other matchings, at most 2ℓ neighbours of v_j are in $V(M_i)$, and M_i has at most ℓ colours. Thus, there will be at least $n/20 - 4\ell > 0$ choices for x. This shows that the greedy process can extend the matchings as described.

Now, greedily, for each M_i, add s edges to M_i by adding an edge of each colour C_j such that M_i remains rainbow and the matchings $M_1, \ldots, M_\ell$ remain edge-disjoint. Similarly to the argument above, we will always have at least $n/20 - 3\ell > 0$ choices to add an edge of each colour C_j to each M_i. The resulting matchings then satisfy the lemma. □

Proof of Lemma 2.7. Let $\ell = \alpha^2 n/10^6$. Let $C_1, \ldots, C_s$ be the α-rich colours of K_n, so that $s \geq n - \alpha n/150$. Let $R \subset K_n$ be the graph with edges without an α-rich colour. Let $r = \max\{n - 1 - s, 0\} \leq \alpha n/150$. As $\delta(R) \geq r$, by Lemma 5.2, we can find ℓ edge-disjoint rainbow matchings with r edges, $M_1, \ldots, M_\ell$ say. For each $1 \leq i \leq \ell$, let $Z_i \subseteq V(M_i)$ contain an arbitrary vertex from each edge in M_i, let $X_i = V(K_m) \setminus Z_i$ and let U_i be the colours of K_n which do not appear in M_i.

Let $G = K_n - M_1 - \ldots - M_\ell$, so that $\delta(G) \geq n - 1 - \ell$. For each $1 \leq i \leq \ell$, let $G_i = G_{U_i}[X_i]$. Each α-rich colour in K_n has all but at most $|Z_i| \leq \alpha n/150$ of its edges in $K_n - Z_i$, and, given any ℓ rainbow forests in G_i, at most $\ell \leq \alpha n/4$ edges of each colour can appear in the forests. Thus, G_i is $(\alpha n/2, \ell)$-robustly-coloured.

Furthermore, G_i has minimum degree at least $n - 1 - \ell - 2|Z_i| \geq (1 - \alpha/60)n$. Therefore, by Lemma 4.1, G contains edge-disjoint rainbow trees $T_1, \ldots, T_\ell$ so that T_i is a spanning tree of G_i. Then, $M_i \cup T_i$, $1 \leq i \leq \ell$, is a collection of ℓ edge-disjoint rainbow spanning trees in K_n. $\quad\square$

References

[1] S. Akbari, A. Alipour, Multicolored trees in complete graphs, *J. Graph Theory*, 54, (2007), 221–232.

[2] R. Brualdi, S. Hollingsworth, Multicolored trees in complete graphs, *J. Combin. Theory Ser. B*, 68, (1996), 310–313.

[3] J. Carraher, S. Hartke, P. Horn, Edge-disjoint rainbow spanning trees in complete graphs, *European J. Combin.*, 57, (2016), 71–84.

[4] G. Constantine, Multicolored parallelisms of isomorphic spanning trees, *Discrete Math. Theor. Comput.* Sci., 5, (2002), 121–126.

[5] P. Horn, Rainbow spanning trees in complete graphs colored by matchings, *preprint*, (2013).

[6] A. Kaneko, M. Kano, K. Suzuki, Three edge-disjoint multicolored spanning trees in complete graphs, *preprint*, (2002).

[7] A. Pokrovskiy, B. Sudakov, Linearly many rainbow trees in properly edge-coloured complete graphs, *arXiv preprint arXiv:1703.07301*, (2017).

[8] A. Schrijver, Combinatorial optimization. Polyhedra and efficiency. Vol. B, volume 24 of Algorithms and Combinatorics. Springer-Verlag, Berlin, 2003. Matroids, trees, stable sets, Chapters 39–69.

[9] K. Suzuki, A necessary and sufficient condition for the existence of a heterochromatic spanning tree in a graph. *Graphs Combin.*, 22(2):261–269, 2006.