On the number of K_4-saturating edges

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Abstract

Let G be a K_4-free graph, an edge in its complement is a K_4-*saturating* edge if the addition of this edge to G creates a copy of K_4. Erdős and Tuza conjectured that for any n-vertex K_4-free graph G with $\lfloor n^2/4 \rfloor + 1$ edges, one can find at least $(1+o(1))\frac{n^2}{16}$ K_4-saturating edges. We construct a graph with only $\frac{2n^2}{33}$ K_4-saturating edges. Furthermore, we prove that it is best possible, i.e., one can always find at least $(1+o(1))\frac{2n^2}{33}$ K_4-saturating edges in an n-vertex K_4-free graph with $\lfloor n^2/4 \rfloor + 1$ edges.

1 Introduction

The notation in this paper is standard. For a graph G, denote by $\overline{G}$ its complement. For any vertex $v \in V(G)$ and vertex subsets $U, W \subseteq V(G)$, denote $N(v) := \{u : uv \in E(G)\}$, $N(U) := \bigcap_{v \in U} N(v)$ and $E(U, W)$ the set of cross edges between U and W.

Mantel [14] showed that the maximum number of edges in an n-vertex triangle-free graph is $\lfloor n^2/4 \rfloor$. Rademacher (unpublished) extended this result by showing that any n-vertex graph with $\lfloor n^2/4 \rfloor + t$ edges contains at least $t\lfloor n/2 \rfloor$ triangles, for $t = 1$. Lovász and Simonovits [13], improving Erdős [6], proved this for every $t \leq n/2$. Erdős [7] showed analogue results for cliques, and Mubayi [15, 16] proved relevant results for color-critical graphs and some hypergraphs.

In general, we call Erdős-Rademacher-type problem the following: for any extremal question, what is the number of forbidden configurations appearing in a graph somewhat denser than the extremal graph? This type of problems have been studied in various contexts: A *book* of size q consists of q triangles sharing a common edge. Khadžiivanov and Nikiforov [11], answering a question of Erdős, showed that any n-vertex graph with $\lfloor n^2/4 \rfloor + 1$ edges contains a book of size at least $n/6$. In the context of Sperner's Theorem, Kleitman [12], answering a question of Erdős and Katona, determined the minimum number of 2-chains in a poset whose

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size is larger than its largest anti-chain. Recently, this theorem was extended to k-chains by Das, Gan and Sudakov [5].

Let G be an n-vertex K_4-free graph, an edge in $\overline{G}$ is a K_4-*saturating* edge if the addition of this edge to G creates a copy of K_4. Denote by $f(G)$ the number of K_4-saturating edges in $\overline{G}$ and by $f(n,e)$ the maximum integer ℓ such that every n-vertex K_4-free graph with e edges must have at least ℓ K_4-saturating edges. The first extremal result related to clique-saturating edges was by Bollobás [4] who proved that if every edge in $\overline{G}$ is a K_r-saturating edge, then $e(G) \geq \binom{n}{2} - \binom{n-r+2}{2}$ and this bound is best possible. Later it was extended by Alon [1], Frankl [9] and Kalai [10] using linear algebraic method. Recently, saturation problems were phrased in the language of 'graph bootstrap percolation', see [2] and [3].

In the case of K_4, Bollobás' example is the following: let F be an n-vertex K_4-free graph with two vertices adjacent to all other vertices which form an independent set. This graph has only $2n-3$ edges, and yet all edges in $\overline{F}$ are K_4-saturating edges. To the other extreme, $K_{\lceil n/2\rceil,\lfloor n/2\rfloor}$ shows that a graph could have up to $\lfloor n^2/4\rfloor$ edges with no K_4-saturating edge, i.e. $f(n,\lfloor n^2/4\rfloor)=0$. Erdős and Tuza [8] conjectured that if a K_4-free graph G has $\lfloor n^2/4\rfloor+1$ edges, then suddenly there are at least $(1+o(1))n^2/16$ K_4-saturating edges. They also stated, without giving any specific example, that there is a graph with at most $(1+o(1))n^2/16$ K_4-saturating edges. Our guess is the following: add a new vertex and make it adjacent to roughly half of vertices in each partite set of $K_{\lceil n/2\rceil-1,\lfloor n/2\rfloor}$. This conjecture can be considered, as formulated before, an Erdős-Rademacher-type problem concerning the number of K_4-saturating edges.

Conjecture 1.1 (Erdős-Tuza [8])**.**

$$f(n,\lfloor n^2/4\rfloor+1)=(1+o(1))\frac{n^2}{16}.$$

We disprove this conjecture. We give a counterexample with only $\frac{2n^2}{33}$ K_4-saturating edges. Furthermore, we prove that $(1+o(1))\frac{2n^2}{33}$ is best possible, that is, one can always find at least $(1+o(1))\frac{2n^2}{33}$ K_4-saturating edges in an n-vertex K_4-free graph with $\lfloor n^2/4\rfloor+1$ edges.

Theorem 1.2. *For $n \geq 73$,*

$$\frac{2n^2}{33}-\frac{3n}{11} \leq f(n,\lfloor n^2/4\rfloor+1) \leq \frac{2n^2}{33}-\frac{7n}{33}.$$

We shall prove the following theorem, which implies the lower bound in Theorem 1.2.

Theorem 1.3. *Let G be an n-vertex K_4-free graph with $\lfloor n^2/4\rfloor$ edges, for $n \geq 73$. If G contains a triangle, then*

$$f(G) \geq \frac{2n^2}{33}-\frac{3n}{11}.$$

This is best possible when n is divisible by 66.

Proof of Theorem 1.2. The upper bound is by the construction described in Section 2. For the lower bound, let G be a K_4-free graph with $\lfloor n^2/4 \rfloor + 1$ edges. By Mantel's theorem, it contains a triangle. Let G' be a subgraph obtained from G by removing an edge such that G' contains a triangle. By Theorem 1.3, $f(G') \geq \frac{2n^2}{33} - \frac{3n}{11}$. The relation $f(G) \geq f(G')$ completes the proof. $\square$

Remark: (i) A slight modification of our proof gives the following stability result: Given any K_4-free graph G with $(1 - o(1))n^2/4$ edges, if G contains a triangle, then $f(G) \geq (1 - o(1))2n^2/33$.

(ii) Unlike the case about the number of triangles in [6] and [13], where every additional edge, up to $n/2$, gurantees $\lfloor n/2 \rfloor$ additional triangles, in our problem, even with linear many extra edges, the number of K_4-saturating edges is still at most $(1 + o(1))2n^2/33$. In particular, $f\left(n, \lfloor \frac{n^2}{4} \rfloor + t\right) = \frac{2n^2}{33} + O(n)$ for $1 \leq t \leq \frac{n}{66}$.

(iii) One might define a K_r-saturating edge of a graph G, for $r \geq 5$, as we did for K_4. Denote by $\mathrm{ex}(n, K_{r-1})$ the maximum size of an n-vertex K_{r-1}-free graph. We think that a similar phenomenon holds: if G is K_r-free and $e(G) = \mathrm{ex}(n, K_{r-1}) + 1$, then the number of K_r-saturating edges is at least $\left(\frac{2(r-3)^2}{(r-1)(4r^2-19r+23)} + o(1)\right) n^2$. A generalization of our construction shows that if the conjecture is true, then it is best possible. (The construction is an appropriate blow-up of the following graph: take a new vertex and make it adjacent to exactly one vertex in each partite set of a $(r-2)$-partite complete graph $K_{2,\ldots,2}$.) Some of the ideas of our proof works for $r \geq 5$ as well, but some does not.

The paper is organized as follows: We give a construction for the upper bound in Theorem 1.2 and an extremal example for Theorem 1.3 in Section 2. The proof for Theorem 1.3 is given in Section 3. We will omit floors and ceilings when it is not critical and we make no effort optimizing some of the constants.

2 Upper bound constructions

Fix an integer n divisible by 66. We present an n-vertex K_4-free graph H with $n^2/4 + n/66$ edges and $f(H) = 2n^2/33 - 7n/33$. Note that from this graph one can easily remove $n/66 - 1$ edges without changing the number of K_4-saturating edges. We also give an extremal example showing the bound in Theorem 1.3 is best possible.

Construction for Theorem 1.2: To construct H, start with a C_5 on $\{v_1, v_2, v_3, v_4, v_5\}$ with a chord v_1v_3. Blow up each v_i to an independent set V_i of the following size: $|V_1| = |V_3| = 16n/66$, $|V_2| = 4n/66 + 1$, $|V_4| = 15n/66$ and $|V_5| = 15n/66 - 1$, see Figure 1. Then H is K_4-free with $n^2/4 + n/66$ edges. The only K_4-saturating edges are those in V_1, V_2, V_3, which gives $f(H) = 2n^2/33 - 7n/33$.

Construction for Theorem 1.3: Define H' the same way as H, except that $|V_2'| = 4n/66$ and $|V_4'| = 15n/66$. This graph is K_4-free with $n^2/4$ edges and $f(H') = \frac{2n^2}{33} - \frac{3n}{11}$.

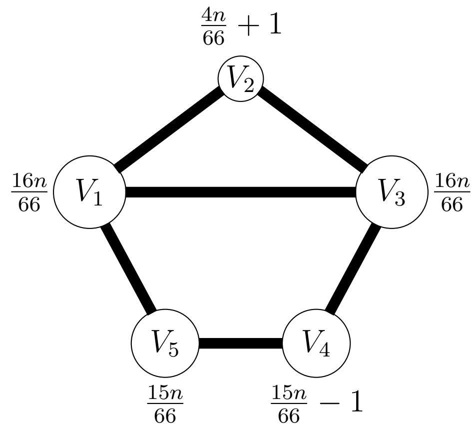

Figure 1: A K_4-free graph H with $e(H)=\frac{n^2}{4}+\frac{n}{66}$ and $f(H)=\frac{2n^2}{33}-\frac{7n}{33}$.

3 Proof of Theorem 1.3

Let G be a K_4-free graph with $n^2/4$ edges and containing a triangle. Fix, in G, a maximum family of vertex-disjoint triangles, say $\mathcal{T}=\{T_1,T_2,...,T_{tn}\}$, where $0<t\le 1/3$. We write $V(\mathcal{T})$ for $\bigcup_{i=1}^{tn}V(T_i)$, $E(\mathcal{T})$ for $E(G[V(\mathcal{T})])$ and $e(\mathcal{T}):=|E(\mathcal{T})|$. Let $G'=G-V(\mathcal{T})$, since $\mathcal{T}$ is of maximum size, G' is a K_3-free graph with $e(G')\le\frac{(1-3t)^2n^2}{4}$. Denote by r_1n^2 the number of K_4-saturating edges incident to $V(\mathcal{T})$, and by r_2n^2 the number of K_4-saturating edges in $V(G')$. Hence $f(G)=(r_1+r_2)n^2$. First we give a lower bound on r_1.

Lemma 3.1.

$$r_1n^2\ge\left(\frac{1}{4}-t+\frac{3t^2}{2}\right)n^2-e(G')-\frac{3}{2}tn\ge\left(\frac{t}{2}-\frac{3t^2}{4}\right)n^2-\frac{3}{2}tn.$$

Proof. Let $t_i=e(T_i,G\setminus\bigcup_{j=1}^{i}T_j)$, clearly $\sum_{i=1}^{tn}t_i=e(G)-e(G')-3tn$. Since G is K_4-free, every vertex can have at most two neighbors in each triangle. Thus $t_i-(n-3i)$ is a lower bound on the number of vertices in $G\setminus\bigcup_{j=1}^{i}T_j$ having degree 2 in T_i, each of which gives a K_4-saturating edge. Indeed, say $V(T_1)=\{x,y,z\}$, and $w\in N(x)\cap N(y)$, then wz is a K_4-saturating edge. Thus,

$$\begin{aligned} r_1n^2 &\ge \sum_{i=1}^{tn}(t_i-(n-3i))=(e(G)-e(G')-3tn)-\left(tn^2-3\frac{tn(tn+1)}{2}\right)\\ &\ge \left(\frac{1}{4}-t+\frac{3t^2}{2}\right)n^2-e(G')-\frac{3}{2}tn\ge\left(\frac{t}{2}-\frac{3t^2}{4}\right)n^2-\frac{3}{2}tn, \end{aligned}$$

where the last inequality follows from $e(G')\le\frac{(1-3t)^2n^2}{4}$. $\square$

Let $T_i\in\mathcal{T}$ be a triangle in $\mathcal{T}$. Denote by $N_j(T_i)\subseteq V(G')$, for $0\le j\le 3$, the set of vertices in G' that has exactly j neighbors in T_i. Since G is K_4-free, $N_3(T_i)=\emptyset$, for every T_i's. Further define $p_0(T_i) = \frac{|N_0(T_i)|}{n}$, $p_1(T_i) = \frac{|N_1(T_i)|}{n}$ and $p_2(T_i) = \frac{|N_2(T_i)|}{n}$. Thus by definition, $p_0(T_i) + p_1(T_i) + p_2(T_i) = 1 - 3t$.

The next lemma shows that there is a triangle $T \in \mathcal{T}$ with large $|N_2(T)|$.

Lemma 3.2. *There exists a triangle $T \in \mathcal{T}$, such that*
(i) $e(T, G') \geq \left(\frac{3}{2} - \frac{21t}{4}\right) n$, and
(ii) $p_2(T) \geq \frac{1}{2} - \frac{9t}{4} + p_0(T)$.

Proof. (i) The edge set of G can be partitioned into $E(G'), E(\mathcal{T}, G')$ and $E(\mathcal{T})$. Notice that since G is K_4-free, there are at most 6 edges between any pair of triangles in $\mathcal{T}$. Hence $e(\mathcal{T}) \leq 3tn + 6\binom{tn}{2} = 3t^2n^2$.

Thus we have $e(\mathcal{T}, G') = e(G) - e(G') - e(\mathcal{T}) \geq \frac{n^2}{4} - \frac{(1-3t)^2n^2}{4} - 3t^2n^2 \geq \left(\frac{3t}{2} - \frac{21t^2}{4}\right) n^2$. Therefore, there exists a triangle $T \in \mathcal{T}$ with $e(T, G') \geq e(\mathcal{T}, G')/(tn) \geq \left(\frac{3}{2} - \frac{21t}{4}\right) n$.
(ii) Let $T \in \mathcal{T}$ be a triangle satisfying (i). Note that $2p_2(T) + p_1(T) = \frac{e(T,G')}{n}$. Using $p_0(T) + p_1(T) + p_2(T) = 1 - 3t$, we have $p_2(T) - p_0(T) \geq \frac{3}{2} - \frac{21t}{4} - (1 - 3t) = \frac{1}{2} - \frac{9t}{4}$. $\square$

From now on, we let $T = \{x, y, z\}$ be a triangle in $\mathcal{T}$ sending the most edges to G', hence it has the two properties of Lemma 3.2. For brevity we write $p_j = p_j(T)$ and $N_i = N_i(T)$ for $0 \leq j \leq 2$. Furthermore, define $A = N_{G'}(xy)$, $B = N_{G'}(yz)$ and $C = N_{G'}(xz)$. Note that A, B, C are pairwise disjoint independent sets, otherwise $T \cup A \cup B \cup C$ contains a copy of K_4. Define $N_x := N_{G'}(x), N_y := N_{G'}(y)$ and $N_z := N_{G'}(z)$. Let $a = \frac{|A|}{|N_2|}$, $b = \frac{|B|}{|N_2|}$ and $c = \frac{|C|}{|N_2|}$, thus $a + b + c = 1$. For $1 \leq k \leq 3$, we say that T spans a *k-joint-book*, if among A, B, C, exactly $3 - k$ of them are empty sets.

Lemma 3.3. *If T spans a 3-joint-book, then we have*

$$r_2n^2 \geq \frac{1}{6}\left[\frac{3}{2} - \frac{21t}{4}\right]^2 n^2 - e(\overline{G'}) - (1 - 3t)n.$$

Proof. First notice that N_x, N_y and N_z are all independent sets. Indeed, suppose N_x contains an edge, then $T \cup N_x \cup B$ contains two vertex-disjoint triangles, contradicting the maximality of $\mathcal{T}$.

Note that $\binom{|N_x|}{2} + \binom{|N_y|}{2} + \binom{|N_z|}{2} \leq r_2n^2 + e(\overline{G'})$. Indeed, every pair of vertices in N_x, N_y or N_z gives a non-edge in G' and those K_4-saturating edges in A, B, C are counted twice. Additionally, $|N_x| + |N_y| + |N_z| = e(T, G') \geq \left(\frac{3}{2} - \frac{21t}{4}\right) n$, and $e(T, G') \leq 2(1 - 3t)n$. Thus,

$$\begin{aligned} r_2n^2 + e(\overline{G'}) &\geq \binom{|N_x|}{2} + \binom{|N_y|}{2} + \binom{|N_z|}{2} \geq 3\binom{e(T, G')/3}{2} \\ &= \frac{1}{6}(e(T, G'))^2 - \frac{1}{2}e(T, G') \geq \frac{n^2}{6}\left[\frac{3}{2} - \frac{21t}{4}\right]^2 - (1 - 3t)n. \end{aligned}$$

$\square$

We first show that if T spans a 3-joint-book, then $f(G) \geq 2n^2/33 - 3n/11$.

Lemma 3.4. *For $n \geq 73$, if T spans a 3-joint-book, then $f(G) \geq \frac{2n^2}{33} - \frac{3n}{11}$.*

Proof. Note that $e(G') + e(\overline{G'}) = \frac{(1-3t)^2n^2}{2} - \frac{(1-3t)n}{2}$. By Lemmas 3.1 and 3.3, we have

$$\begin{aligned} f(G) &= (r_1 + r_2)n^2 \geq \left(\frac{1}{4} - t + \frac{3t^2}{2}\right)n^2 - e(G') - \frac{3}{2}tn \\ &+ \frac{1}{6}\left[\frac{3}{2} - \frac{21t}{4}\right]^2 n^2 - e(\overline{G'}) - (1-3t)n \\ &\geq \left(\frac{51t^2}{32} - \frac{5t}{8} + \frac{1}{8}\right)n^2 - \frac{n}{2} \geq \frac{13n^2}{204} - \frac{n}{2} \geq \frac{2n^2}{33} - \frac{3n}{11}, \end{aligned}$$

since $\frac{51t^2}{32} - \frac{5t}{8} + \frac{1}{8} \geq \frac{13}{204}$ when $0 < t \leq 1/3$, and the last inequality holds for $n \geq 73$. $\square$

Proof of Theorem 1.3. By Lemma 3.4, we may assume that T spans a k-joint-book with $k \leq 2$. Without loss of generality assume that $B = \emptyset$, i.e. $b = 0$. Then $a + c = 1$ and $|A| + |C| = p_2n$. Notice that each pair of vertices in A and C is a K_4-saturating edge, hence

$$r_2n^2 \geq \binom{|A|}{2} + \binom{|C|}{2} \geq 2\binom{p_2n/2}{2} = \frac{p_2^2}{4}n^2 - \frac{p_2n}{2}. \tag{1}$$

If $t \geq \frac{1}{5}$, then Lemma 3.1 implies $f(G) \geq r_1n^2 \geq \left(\frac{t}{2} - \frac{3t^2}{4}\right)n^2 - \frac{n}{2} \geq \frac{2n^2}{33}$ for $n \geq 54$. Thus we may assume that $t < \frac{1}{5}$. The right hand side in (1) is minimized when p_2 is at its lower bound provided by Lemma 3.2, as $\frac{1}{2} - \frac{9t}{4} > \frac{1}{n}$ for $n \geq 20$. Hence

$$r_2n^2 \geq \frac{1}{4}\left(\frac{1}{2} - \frac{9t}{4}\right)^2 n^2 - \frac{1}{2}\left(\frac{1}{2} - \frac{9t}{4}\right)n.$$

Therefore using Lemma 3.1, we have

$$\begin{aligned} f(G) &= (r_1 + r_2)n^2 \geq \left(\left(\frac{t}{2} - \frac{3t^2}{4}\right) + \frac{1}{4}\left(\frac{1}{2} - \frac{9t}{4}\right)^2\right)n^2 - \frac{1}{2}\left(3t + \frac{1}{2} - \frac{9t}{4}\right)n \\ &= \left(\frac{33t^2}{64} - \frac{t}{16} + \frac{1}{16}\right)n^2 - \frac{1}{2}\left(\frac{3t}{4} + \frac{1}{2}\right)n \geq \frac{2n^2}{33} - \frac{3n}{11} - \frac{3}{44}, \end{aligned}$$

where the function on the right hand side is minimized at $t = \frac{2}{33} + \frac{4}{11n}$. Since both tn and $f(G)$ are integers, checking all n modulo 33, we have

$$f(G) \geq \frac{2n^2}{33} - \frac{3n}{11}.$$

We remark that the extremal example corresponds to the last case when $t < 1/5$ and T spans a 2-joint book with $|A| = |C|$. $\square$

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