Clique immersion in graphs without fixed bipartite graph

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Abstract

A graph G contains H as an *immersion* if there is an injective mapping $\phi : V(H) \to V(G)$ such that for each edge $uv \in E(H)$, there is a path P_{uv} in G joining vertices $\phi(u)$ and $\phi(v)$, and all the paths P_{uv}, $uv \in E(H)$, are pairwise edge-disjoint. An analogue of Hadwiger's conjecture for the clique immersions by Lescure and Meyniel states that every graph G contains $K_{\chi(G)}$ as an immersion. We consider the average degree condition and prove that for any bipartite graph H, every H-free graph G with average degree d contains a clique immersion of order $(1-o(1))d$, implying that Lescure and Meyniel's conjecture holds asymptotically for graphs without fixed bipartite graph.

1 Introduction

A graph H is a *minor* of another graph G if H can be obtained from G via vertex deletions, edge deletions and edge contractions. A conjecture of Hadwiger [8] states that every graph G with $\chi(G) \geq t$ contains K_t as a minor. This conjecture is widely open for $\chi(G) \geq 7$; the case $\chi(G) = 5$ is equivalent to the celebrated Four-Color Theorem and the case $\chi(G) = 6$ was solved by Robertson, Seymour and Thomas [21]. A graph H is a *topological minor* of another graph G if there is an injective mapping $\phi : V(H) \to V(G)$ such that for each edge $uv \in E(H)$, there is a path P_{uv} in G joining vertices $\phi(u)$ and $\phi(v)$, and all the paths P_{uv}, $uv \in E(H)$, are pairwise internally vertex-disjoint. A stronger conjecture proposed by Hajós in 1940's [9] states that every graph G contains $K_{\chi(G)}$ as a topological minor. However, this conjecture is known to be false in general: Catlin [2] disproved this conjecture for all $\chi(G) \geq 7$. It is also natural to consider graphs with given average degree. In this direction, Kostochka [12] and independently, Thomason [23] proved that average degree $d(G) = \Omega(t\sqrt{\log t})$ in a graph G forces K_t as a minor, and this bound is optimal. This remained the best order of magnitude for Hadwiger's conjecture until very recent breakthroughs by Norin, Postle and Song [19] and Postle [20].

In this paper, we consider immersions, first introduced by Nash-William [18]. We say G contains H as an *immersion* if there exists an injective mapping $\phi : V(H) \to V(G)$ such that for each edge $uv \in E(H)$, there is a path P_{uv} in G connecting $\phi(u)$ and $\phi(v)$; and all the paths P_{uv}, $uv \in E(H)$, are pairwise edge-disjoint. We call the vertices $\{\phi(v) \mid v \in V(H)\}$ the *branch* vertices of the immersion. As a weakening of topological minor, immersion relation requires paths to be pairwise edge-disjoint rather than vertex-disjoint. Although graph minors and graph immersions are incomparable, Robertson and Seymour [22] showed that graphs are well-quasi-ordered by immersion, analogous to their celebrated graph minors project. An immersion variant of Hadwiger's conjecture was proposed by Lescure and Meyniel [16] in 1989, and independently, by Abu-Khzam and Langston [1] in 2003.

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Conjecture 1.1. [1, 16] *Every graph G with $\chi(G) \geq t$ contains K_t as an immersion.*

Conjecture 1.1 has seen more success than the Hadwiger's Conjecture: the cases $t \leq 4$ are trivial, while the cases $5 \leq t \leq 7$ are proved by DeVos, Kawarabayashi, Mohar and Okamura [5], who proved that a lower bound on the minimum degree suffices to ensure K_t as an immersion. That is, $f(t) = t - 1$ holds for any $t \in \{5, 6, 7\}$, where $f(t)$ is the least integer such that every graph G with $\delta(G) \geq f(t)$ contains K_t as an immersion. It is easy to see that $f(t) \geq t - 1$. For $t \geq 8$, there are infinitely many constructions [3, 4] showing that $f(t) \geq t$. The first linear upper bound is due to DeVos, Dvořák, Fox, McDonald, Mohar and Scheide [4], showing that $f(t) \leq 200t$, and it was then improved to $11t + 7$ by Dvořák and Yepremyan [6], who asked whether for all $t \geq 8$, $f(t) = t$. Building on the work of [6], Gauthier, Le and Wollan [7] showed that $f(t) \leq 7t + 7$.

Our work is motivated by a result of Kühn and Osthus [15]. They proved that, for any fixed bipartite graph H, Hadwiger's conjecture holds strongly for any H-free graph G. In fact, in such G, they found a clique minor of order polynomially larger than $d(G)$. Improved bounds on the order of clique minor in H-free graphs for bipartite H were later obtained by Krivelevich and Sudakov [14] and Norin, Postle and Song [19].

Our main result reads as follows. It in particular implies that Conjecture 1.1 is asymptotically true if we forbid any bipartite graph H.

Theorem 1.2. *Let H be a bipartite graph containing a cycle. Then for any positive constant ε, there exists $d_0 = d_0(\varepsilon, H)$ such that every H-free graph G with $d(G) = d \geq d_0$ contains a clique immersion of order $(1 - \varepsilon)d$.*

The bound above is asymptotically optimal as G could be d-regular. It would be interesting to improve on the additive error term.

Our approach differs from previous work on immersions in [4, 5, 6, 7]. Our proof makes use of certain expander and builds on the techniques developed in the work of Liu and Montgomery [17] on embedding clique subdivisions. The bulk of the work is to handle sparse expanders. In this case, it is easy to find clique immersion of order $\Omega(d)$. To embed $K_{(1-o(1))d}$-immersion in sparse expanders, we need to find vertices that can expand past a relatively large set of vertices. Inspired by the work of Halsegrave, Kim and Liu [10], we make iterative use of the expanders to find such vertices. Roughly speaking we grow a vertex in its correpsonding subexpander robustly till it reaches large enough size to enjoy further expansion in the main expander.

The rest of the paper will be organized as follows. In Section 2, we will introduce some necessary results and tools we need for our proof. The proof of our main result is then given in Sections 3 and 4.

2 Preliminaries

For a set of vertices $X \subseteq V(G)$, denote its *external neighbourhood* by $N_G(X) := \{u \in V(G) \setminus X : uv \in E(G) \text{ for some } v \in X\}$. Furthermore, denote by $\partial_G(X)$ the *edge boundary* of X, i.e. $E_G[X, V(G) \setminus X]$. Define $G - X$ to be the induced subgraph of G on $V(G) \setminus X$ and for a subgraph F, use $G \setminus F$ to denote the spanning subgraph with $E(F)$ removed. For two sets $A, B \subseteq V(G)$, a path P is an (A, B)-path if P connects u, v for some $u \in A, v \in B$. Moreover, the *distance* between A and B is the minimum length of an (A, B)-path. For each $r \in \mathbb{N}$, the *r-th sphere* around X, denoted by $N_G^r(X)$, is the set of vertices with distance exactly r from X. So $N_G^0(X) = X$ and $N_G^1(X) = N_G(X)$. Denote by $B_G^r(X)$ the *ball* of radius r around X, i.e. $B_G^r(X) = \cup_{0 \leq i \leq r} N_G^i(X)$. Throughout the proof, all logarithms are in the natural basis.

As any bipartite graph is a subgraph of a complete bipartite graph, for Theorem 1.2, we may assume that our graph G is $K_{s,t}$-free for some $2 \leq s \leq t$. Writting $d = d(G)$, the following bound follows from the classical result of Kővári, Sós and Turán [13]:

$$n/d = \Omega(d^{\frac{1}{s-1}}). \tag{1}$$

We will also make use of the following bipartite version.

Lemma 2.1. *Let $2 \le s \le t$. Then there exists a constant c such that the following holds. Let $G = (V_1, V_2, E)$ be a $K_{s,t}$-free bipartite graph with $|V_1| = n_1, |V_2| = n_2$. Then*

$$e(G) \le c n_1^{1-1/s} n_2.$$

2.1 Robust sublinear expander

For $\varepsilon_1 > 0$ and $k > 0$, let $\rho(x)$ be the function

$$\rho(x) = \rho(x, \varepsilon_1, k) := \begin{cases} 0 & \text{if } x < k/5 \\ \varepsilon_1/\log^2(15x/k) & \text{if } x \ge k/5, \end{cases} \tag{2}$$

where, when it is clear from context we will not write the dependency on ε_1 and k of $\rho(x)$. Note that $\rho(x) \cdot x$ is increasing for $x \ge k/2$. In [11], Komlós and Szemerédi introduced a notion of *(ε_1, k)-expander G* in which for any subset $X \subseteq V(G)$ with $k/2 \le |X| \le |V(G)|/2$, we have $|N_G(X)| \ge \rho(|X|) \cdot |X|$. In this paper, we shall utilize the following robust version recently developed by Haslegrave, Kim and Liu [10], such that similar expansion property occurs even after removing a relatively small set of edges.

Definition 2.2. A graph G is an *(ε_1, k)-robust-expander* if for all subsets $X \subseteq V(G)$ of size $k/2 \le |X| \le |V(G)|/2$ and any subgraph $F \subseteq G$ with $e(F) \le d(G) \cdot \rho(|X|)|X|$, we have.

$$|N_{G\setminus F}(X)| \ge \rho(|X|) \cdot |X|. \tag{3}$$

We will use the following version of expander lemma in [10], which states that every graph contains a robust expander with almost the same average degree.

Lemma 2.3. [10] *Let $C > 30$, $0 < \varepsilon_1 \le \frac{1}{10C}, 0 < \varepsilon_2 < 1/2, d > 0, \eta = C\varepsilon_1/\log 3$ and $\rho(x) = \rho(x, \varepsilon_1, k)$ be as in* (2)*. Then every graph G has a subgraph G' that is an (ε_1, k)-robust-expander with $d(G') \ge (1-\eta)d(G)$ and $\delta(G') \ge d(G')/2$.*

The following small diameter property is the key property of the expanders that we will repeatedly make use of. It roughly says that we can find a relatively short path between two large sets, avoiding a small set of vertices and edges.

Lemma 2.4 (Robust small diameter, Lemma 2.3 in [10])**.** *Let $0 < \varepsilon_1, \varepsilon_2 < 1$ and G be an n-vertex $(\varepsilon_1, \varepsilon_2 d)$-robust-expander. Let X_1, X_2 be two sets of size $x \ge \varepsilon_2 d/2$, Y be a vertex set of size at most $\rho(x)x/4$ and F be a subgraph with at most $d(G)\rho(x)x$ edges. Then there is an (X_1, X_2)-path of length at most $\frac{2}{\varepsilon_1}\log^3(15n/\varepsilon_2 d)$ in $(G \setminus F) - Y$.*

The following is our main lemma, which finds in a robust expander a clique immersion of asymptotically optimal size.

Lemma 2.5. *Let $0 < \varepsilon_1 \le 1/400, 0 < \varepsilon_2 < 1/2, \eta \ge \max\{\frac{40\varepsilon_1}{\log 3}, 5\varepsilon_2\}$ and $2 \le s \le t$. Then there exists d_0 satisfying the following. Let G be a $K_{s,t}$-free $(\varepsilon_1, \varepsilon_2 d)$-robust-expander of order n and $d(G) = d \ge d_0$. Then G contains a clique immersion of order at least $(1-9\eta)d(G)$.*

Theorem 1.2 immediately follows from Lemmas 2.3 and 2.5.

Proof of Theorem 1.2. Let $C = 40$, $\varepsilon_1 = \varepsilon \log 3/400$, $\varepsilon_2 \le \varepsilon/50$ and $\eta = \varepsilon/10$. Then $\eta \ge \eta' := 40\varepsilon_1/\log 3$ and by Lemma 2.3 with $\varepsilon_1, \varepsilon_2, C = 40, \eta'$, G contains a subgraph G' that is an $(\varepsilon_1, \varepsilon_2 d)$-robust-expander with $d(G') \ge (1-\eta')d(G) \ge (1-\eta)d(G)$ and $\delta(G') \ge d(G')/2$. Applying then Lemma 2.5 to G' with $\varepsilon_1, \varepsilon_2, \eta = \varepsilon/10$, we obtain a clique immersion of order at least $(1-9\eta)d(G') \ge (1-\varepsilon)d(G)$ in G', which is also an immersion in G. □

3 Proof of Lemma 2.5 dense case: $d \geq \log^{200s} n$

In this section, we prove Lemma 2.5 assuming in addition that $d \geq \log^{200s} n$. Throughout the rest of this paper, we write

$$\ell = (1-5\eta)d, \quad \ell' = (1-4\eta)d, \quad m := \frac{2}{\varepsilon_1}\log^3\left(\frac{15n}{\varepsilon_2 d}\right).$$

Note that by (1), when d is sufficiently large, then n/d and also m are sufficiently large, and

$$n/d \geq m^{200} \quad \text{and} \quad d \geq m^{50s}. \tag{4}$$

Also, for sufficiently large d, since $\rho(x)$ is decreasing, we have that for every $\varepsilon_2 d/2 \leq x \leq n$,

$$\rho(x) \geq \rho(n) \geq \frac{1}{m}. \tag{5}$$

Definition 3.1. Given integers $h_1, h_2, h_3 > 0$, an (h_1, h_2, h_3)-*unit* F is a graph consisting of a center v, h_1 vertex-disjoint stars $S(u_i)$ centered at u_i, each of size h_2, and edge-disjoint (v, u_i)-paths, $i = 1, \dots, h_1$, each of length at most h_3, moreover, the set of interior vertices in all (v, u_i)-paths is disjoint from all leaves in $\bigcup_{i=1}^{h_1} S(u_i)$. By the *exterior* of the unit, denoted by $\mathsf{Ext}(F)$, we mean the set of all leaves in $\bigcup_{i=1}^{h_1} S(u_i)$. We call all (v, u_i)-paths the *branches* of F.

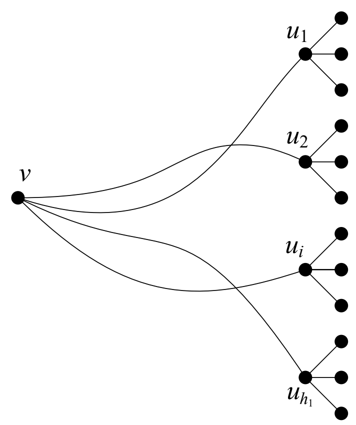

Figure 1: (h_1, h_2, h_3)-unit

The following lemma guarantees a large collection of edge-disjoint units with distinct centers.

Lemma 3.2. *For each $0 < \varepsilon_1, \varepsilon_2 < 1, \eta \geq \max\{\frac{40\varepsilon_1}{\log 3}, 5\varepsilon_2\}$ and $2 \leq s \leq t$, there exists $C > 0$ such that the following holds for all n and d with $d \geq \log^{200s} n$ and $n/d \geq C$. If G is an n-vertex $K_{s,t}$-free $(\varepsilon_1, \varepsilon_2 d)$-robust-expander with $d(G) = d$, then G contains ℓ' pairwise edge-disjoint $(\ell', m^5, 2m)$-units $F_1, \dots, F_{\ell'}$ with distinct centers $v_1, \dots, v_{\ell'}$, where $\ell' = (1-4\eta)d$ and $m = \frac{2}{\varepsilon_1}\log^3\left(\frac{15n}{\varepsilon_2 d}\right)$.*

We first see how to construct a K_ℓ-immersion, $\ell = (1-5\eta)d$, using Lemma 3.2.

Proof of Lemma 2.5 when $d \geq \log^{200s} n$. Let $F_1, \dots, F_{\ell''}$ be the $(\ell', m^5, 2m)$-units guaranteed by Lemma 3.2, with distinct centers $v_1, \dots, v_{\ell''}$, where $\ell'' = (1-4.5\eta)d$. We will connect pairs of the ℓ'' centers in an arbitrary order greedily as follows to obtain a desired clique immersion.

A1 Greedily connect as many pairs (v_i, v_j) of centers as possible through an $(\mathsf{Ext}(F_i), \mathsf{Ext}(F_j))$-path of length at most m.

A2 Let F_i and F_j be the current pair to connect. Avoid using

- edges that are used in previous connections;
- all other centers v_p, $p \neq i, j$;
- edges that are in branches of all units.

A3 In each F_i, a star is *occupied* if a leaf of it was previously used as an endpoint of some $(\mathsf{Ext}(F_i), \mathsf{Ext}(F_j))$-path.

A4 A star in a unit is *over-used* if half of its leaves were used in previous connections. Discard a unit if it has at least $\eta d/4$ over-used stars.

Note that an $(\mathsf{Ext}(F_i), \mathsf{Ext}(F_j))$-path together with the corresponding branches within F_i and F_j form a (v_i, v_j)-path of length at most $6m$. The total number of edges used in all connections is at most $\binom{\ell''}{2} \cdot 6m \leq 3d^2m$, while the total number of edges in branches of all units is at most $2d^2m$. Thus, in each connection, we avoid using a set of at most $5d^2m$ edges and a set of at most d centers $v_i, i \in [\ell'']$ in **A2**.

We claim that there are at least ℓ units survived **A4**, say $F_1, \dots, F_\ell$. Indeed, since all units are edge-disjoint and we discard a unit in **A4** when at least $\eta d/4 \cdot m^5/2$ edges were used before, hence, the number of units discarded is at most

$$\frac{5d^2m}{\eta d \cdot m^5/8} \leq \eta d/2 \leq \ell'' - \ell.$$

We now show that we can connect all pairs of the survived ℓ units. Note that for each survived unit F_i, $i \in [\ell]$, as there are ℓ' branches/stars in each unit, there are at least $\ell' - \ell'' = \eta d/2$ stars not occupied in **A3**, among which there are at least $\eta d/4$ stars that are not over-used. Thus, each v_i can still reach at least

$$\eta d/4 \cdot m^5/2 \geq dm^4 := x$$

vertices in $\mathsf{Ext}(F_i)$. Recall that we avoid at most $5d^2m \leq d(G) \cdot \rho(x) \cdot x$ edges and at most $d \leq \rho(x) \cdot x/4$ vertices. Hence, by Lemma 2.4, we can find the desired $(\mathsf{Ext}(F_i), \mathsf{Ext}(F_j))$-path of length at most m. Finally, notice that extending these paths from the exterior to the corresponding centers of the units yields edge-disjoint (v_i, v_j)-paths, for all $ij \in \binom{[\ell]}{2}$, which yields a desired K_ℓ-immersion. □

3.1 Proof of Lemma 3.2

Firstly, using the $K_{s,t}$-free condition, we show that after deleting a small set of arbitrary vertices, the remaining subgraph still has large average degree.

Claim 3.3. For any subset $Z \subseteq V(G)$ with $|Z| \leq dm^{50}$, we have $d(G - Z) \geq d(G) - \eta d$.

Proof. We may assume that $|Z| \geq \eta d$. Consider the bipartite subgraph $G_1 := G[Z, V(G) \setminus Z]$. Since G_1 is $K_{s,t}$-free, by Lemma 2.1, $e(G_1) \leq c|Z|^{1-1/s}(n - |Z|) \leq c(dm^{50})^{1-1/s}n \leq \frac{cdn}{m^{50/s}}$, for some constant $c > 0$, where the last inequality follows from $d \geq m^{50s}$. Also, $e(G[Z]) \leq \gamma|Z|^{2-1/s} \leq \gamma(dm^{50})^{2-1/s} \leq \frac{\gamma dn}{(dm^{50})^{1/s}}$, for some $\gamma > 0$, where the last inequality follows from (4). Thus

$$d(G - Z) = \frac{nd(G) - 2e(G_1) - 2e(G[Z])}{n - |Z|} \geq d(G) - \frac{2cd}{m^{50/s}} - \frac{\gamma d}{d^{1/s}m^{50/s}} \geq d(G) - \eta d,$$

where the last inequality follows since n/d, hence also m, is sufficiently large. □

Now we are ready to construct the desired units iteratively. Let Z be the centers of all units constructed so far and B be their edge set. Suppose $|Z| < \ell'$. Let $G' := (G - Z) \setminus B$, i.e. the subgraph on vertex set $V(G) \setminus Z$ with edges in B removed. It suffices to find in G' an $(\ell', m^5, 2m)$-unit.

By (4), we have $|B| \leq \ell' \cdot 2dm^5 \leq d^2m^6 \leq \frac{1}{2}\eta d(n - |Z|)$. Together with Claim 3.3, we have

$$d(G') \geq d(G - Z) - 2|B|/(n - |Z|) \geq d(G) - 2\eta d.$$

Similarly, by Claim 3.3, we can iteratively find in G' vertex-disjoint stars $S(v_i)$, $i = 1, \dots, m^{10}$, each of size $d(G) - 3\eta d$ and $S(u_j)$, $j = 1, \dots, dm^{15}$, each of size m^{10}. Indeed, when we delete all the vertices of the stars we found so far, Claim 3.3 guarantees that the remaining subgraph still has large average degree, which allows us to find one more star as required. For each star $S(v_i)$, $i \in \{1, \dots, m^{10}\}$, we use $L(v_i)$ to denote the set of all its leaves. Let $V = \{v_1, \dots, v_{m^{10}}\}$. We will use these vertex-disjoint stars to construct a new $(\ell', m^5, 2m)$-unit in G' as follows.

- Connect as many pairs (v_i, u_j) as possible through an $(L(v_i), u_j)$-path of length at most m, such that there is at most one path between any pair.
- For all connections, avoid using
  - edges used in previous $(L(v_{i'}), u_{j'})$-paths;
  - edges in $\bigcup_{p=1}^{m^{10}} S(v_p)$, $\bigcup_{q=1}^{dm^{15}} S(u_q)$ and B.
  - all vertices in $Z \cup V$.
- For each v_i, a leaf $v \in L(v_i)$ is *occupied* if it is previously used as an endpoint of some $(L(v_i), u_j)$-path.

Claim 3.4. There is a vertex v_i connected to at least $s = (\ell' + \eta d/2)$ distinct centers u_j.

Proof. Suppose to the contrary that each v_i is connected to less than s centers u_j. Then the number of vertices used in all paths is at most $d \cdot m^{10} \cdot m \leq dm^{11}$. Thus, there are at least $dm^{15}/2$ u_j-stars that are completely vertex-disjoint from all those paths, and there are at least $dm^{15}/2 > dm^9 := x$ available centers from u_j-stars, say U'. Inside each v_i-star, there are at least $d(G) - 3\eta d - s = \eta d/2$ leaves not occupied. Thus, there are at least $\eta d/2 \cdot m^{10} > x$ available leaves from v_i-stars, say V'.

Recall that there are at most d^2m^6 edges in B, at most dm^{11} edges used in all paths, at most dm^{10} edges in v_i-stars and at most dm^{25} edges in u_j-stars. Thus, in total, we avoid at most $d^2m^7 \leq d(G) \cdot \rho(x) \cdot x$ edges and at most $|Z| + |V| \leq d + m^{10} \leq \rho(x) \cdot x/4$ vertices. Therefore, by Lemma 2.4, we can find a path of length at most m between U' and V' in G', resulting in one more pair of v_i, u_j connected, a contradiction. □

Let v_i, $u_1, u_2, \cdots, u_s$ be the centers guaranteed by Claim 3.4 with all (v_i, u_j)-paths pairwise edge disjoint. If the set of interior vertices in all (v_i, u_j)-paths is disjoint from $\bigcup_{j=1}^{s} S(u_j)$, then they form a desired unit in G'. Otherwise, we discard a star $S(u_{j'})$ if half of its leaves are used in (v_i, u_j)-paths. We claim that there are at least ℓ' u_j-stars survived, say $S(u_1), \dots, S(u_{\ell'})$. Indeed, recall that all u_j-stars are vertex-disjoint, the number of stars discarded is at most

$$\frac{sm}{m^{10}/2} \leq \eta d/2 \leq s - \ell'.$$

Therefore, each of the ℓ' survived u_j-stars has at least $m^{10}/2 \geq m^5$ leaves that are not used in any (v_i, u_j)-path. These stars, together with the corresponding paths to v_i, form a desired unit in G'.

4 Proof of Lemma 2.5 sparse case: $d < \log^{200s} n$

For the proof of the sparse case, we first deal with a special case when the maximum degree is somewhat bounded, see Lemma 4.4. Then we divide the proof into two cases depending on the number of vertices of sufficiently large degree. In particular, let $Z_1 = \{v \in V(G) \mid d(v) \geq dm^3\}$. Claim 4.5 deals with the case when there exist at least d vertices of degree at least dm^3. If $|Z_1| < d$, then by Claim 4.6 we focus our attention to the subgraph $G' := G - Z_1$ which still has large average degree and additionally small maximum degree. Note, however, that G' might not have the expansion property for small sets. We then show that we can find in G' a collection of small $(\varepsilon_1, \varepsilon_2 d)$-robust-expanders $F_1, F_2, \cdots, F_d$ which are pairwise far apart in G'. Anchoring at these small subexpanders, we will build a desired clique immersion.

4.1 Almost regular sparse expander

In this subsection, we prove Lemma 4.4, which deals with sparse expander that are 'almost' regular. The idea is to choose vertices, say $v_1, v_2, \cdots, v_{\ell'}$, that are pairwise far apart to be the branch vertices of our clique immersion. We grow two balls around each v_i, one inner ball $B^r(v_i)$ and one outer ball $B^{k+r}(v_i)$, where $k = \log n/(800s \log\log n)$ and $r = (\log\log n)^5$. To construct the desired clique immersion, we connect pairs v_i, v_j using a shortest path between the outer balls around them while avoiding all edges in the inner balls of other branch vertices, that is, $\bigcup_{p\neq i,j} E(B^r(v_p))$. Using the robust expansion guaranteed by the Lemma 4.1 and Lemma 4.3, we will be able to regrow new inner and outer balls around v_i to be large enough to enable us to connect v_i to more branch vertices.

First, we need the following lemma, which ensures that a large set of vertices expands well even after deleting a small set of vertices.

Lemma 4.1 (Proposition 3.5 in [10]). *Let $0 < \varepsilon_1 \leq 1/400, 0 < \varepsilon_2 < 1/2$, and G be an n-vertex $(\varepsilon_1, \varepsilon_2 d)$-robust-expander with $d(G) = d$. If $X, Y \subseteq V(G)$ are sets such that $|X| = x \geq \varepsilon_2 d, |Y| \leq \rho(x)x/4$, then for every $i \leq \log n$,*

$$|B^i_{G-Y}(X)| \geq \exp(\sqrt[4]{i}).$$

Definition 4.2. For a set $X \subseteq W$ of vertices, the paths $P_1, \cdots, P_q$ are *edge-disjoint consecutive shortest paths from X within W* in G if the following holds. For each $i \in [q]$, P_i is a shortest path from X to some vertex $v_i \in W$ in the graph $G[W] \setminus \bigcup_{j\in[i-1]} E(P_j)$.

The next expansion lemma ensures that each ball is still large even after deleting a small set of edges.

Lemma 4.3. *Let $0 < \varepsilon_1 \leq 1/400, 0 < \varepsilon_2 < 1/2, \eta \geq \max\{\frac{40\varepsilon_1}{\log 3}, 5\varepsilon_2\}$. Then there exists $d_0 := d_0(\varepsilon_1, \varepsilon_2)$ such that the following holds for all $d \geq d_0$. Suppose G is an n-vertex $(\varepsilon_1, \varepsilon_2 d)$-expander with $d(G) = d$ and K, Z are disjoint sets of vertices with*

$$|K| < n/2,\ |Z| \leq |K|\rho(|K|)/4,\ |N_{G-Z}(K)| \geq \ell' + \varepsilon_2 d,$$

where $\ell' = (1-4\eta)d$. Let $r = (\log\log n)^5$ and $P_1, P_2, \cdots, P_q$ be edge-disjoint consecutive shortest paths from K within $B^r_{G-Z}(K)$ in G such that $q < \ell'$ and denote $E = \bigcup_{j\in[q]} E(P_j)$. Then for any positive integers $t \in [r]$ and D with $D \leq n/2$, we have

$$|B^t_{(G-Z)\setminus E}(K)| \geq \min\left\{D,\ |K| \cdot \left(1 + \frac{\varepsilon_1}{2\log^2(15D/\varepsilon_2 d)}\right)^{t-1}\right\}.$$

Proof. Let $G' = G - Z$. For all $1 \le p \le t-1$, denote by E_p the set of edges in E that go from $B^p_{G'\setminus E}(K)$ to $N_{G'}(B^p_{G'\setminus E}(K))$, then we have

$$N_{G'\setminus E}(B^p_{G'\setminus E}(K)) = N_{G'\setminus E_p}(B^p_{G'\setminus E}(K)).$$

Note that only the first $p+2$ vertices of each P_i can intersect $N_{G'}(B^p_{G'\setminus E}(K))$ for otherwise a shorter path can be found, contradicting the choices $P_1, \ldots, P_q$. Therefore $|E_p \cap E(P_i)| \le p+1$ for each $i \in [q]$ and then so

$$|E_p| \le q(p+1) < \ell'(p+1). \tag{6}$$

Since $|B^p_{G'\setminus E}(K)| \ge |B^1_{G'\setminus E}(K)| \ge |N_{G'}(K)| - q \ge \varepsilon_2 d$, we will show that when d is sufficiently large with respect to $\varepsilon_1, \varepsilon_2$, then for all $p \in [t-1]$,

$$|B^p_{G'\setminus E}(K)| \cdot \rho(|B^p_{G'\setminus E}(K)|) > p+1. \tag{7}$$

Indeed, if this holds, then $|E_p| < \ell'(p+1) < d(G) \cdot |B^p_{G'\setminus E}(K)| \cdot \rho(|B^p_{G'\setminus E}(K)|)$ and it follows from the expansion property as in (3) that

$$|N_{G\setminus E_p}(B^p_{G'\setminus E}(K))| \ge |B^p_{G'\setminus E}(K)| \cdot \rho(|B^p_{G'\setminus E}(K)|).$$

Since $|B^p_{G'\setminus E}(K)| \ge |K|$ and $x\rho(x)$ is increasing in x, we have

$$|Z| \le |K|\rho(|K|)/4 < |B^p_{G'\setminus E}(K)| \cdot \rho(|B^p_{G'\setminus E}(K)|)/2,$$

and so it holds that

$$|N_{G'\setminus E_p}(B^p_{G'\setminus E}(K))| \ge |N_{G\setminus E_p}(B^p_{G'\setminus E}(K))| - |Z| \ge |B^p_{G'\setminus E}(K)| \cdot \rho(|B^p_{G'\setminus E}(K)|)/2. \tag{8}$$

We may assume $|B^p_{G'\setminus E}(K)| \le D$ for all $1 \le p \le t-1$, otherwise we are done. It follows that

$$\begin{aligned}|N_{G'\setminus E}(B^p_{G'\setminus E}(K))| &\ge |B^p_{G'\setminus E}(K)| \cdot \rho(|B^p_{G'\setminus E}(K)|)/2 = |B^p_{G'\setminus E}(K)| \cdot \frac{\varepsilon_1}{2\log^2(\frac{15|B^p_{G'\setminus E}(K)|}{\varepsilon_2 d})} \\ &\ge |B^p_{G'\setminus E}(K)| \cdot \frac{\varepsilon_1}{2\log^2(15D/\varepsilon_2 d)}.\end{aligned}$$

Thus $|B^{p+1}_{G'\setminus E}(K)| \ge |B^p_{G'\setminus E}(K)| \left(1 + \frac{\varepsilon_1}{2\log^2(15D/\varepsilon_2 d)}\right)$ for each $p \in [t-1]$, and then

$$|B^t_{G'\setminus E}(K)| \ge |K| \cdot \left(1 + \frac{\varepsilon_1}{2\log^2(15D/\varepsilon_2 d)}\right)^{t-1}.$$

Now it remains to prove (7), which we will show by induction on p. Let p_0 be the least integer such that for each $p \ge p_0$, we have $p^2/4 \cdot \rho(p^2/4) \ge p+1$. Then $p_0 = O(\sqrt{d})$. The basic cases $1 \le p \le p_0$ easily follows since $|B^p_{G'\setminus E}(K)| \cdot \rho(|B^p_{G'\setminus E}(K)|) \ge |B^1_{G'\setminus E}(K)| \cdot \rho(|B^1_{G'\setminus E}(K)|) > \varepsilon_2 d \cdot \rho(\varepsilon_2 d) > p_0 + 1 \ge p+1$ holds whenever d is sufficiently large. Suppose $p > p_0$, and assume that (7) holds for all p' with $1 \le p' \le p-1$. Then $|E_{p'}| < \ell'(p'+1) < d(G) \cdot |B^{p'}_{G'\setminus E}(K)| \cdot \rho(|B^{p'}_{G'\setminus E}(K)|)$, which together with the expansion property as in (8) implies that

$$\begin{aligned}|B^{p'+1}_{G'\setminus E}(K)| &\ge |B^{p'}_{G'\setminus E}(K)| + |B^{p'}_{G'\setminus E}(K)| \cdot \rho(|B^{p'}_{G'\setminus E}(K)|)/2 \\ &\ge |B^{p'}_{G'\setminus E}(K)| + (p'+1)/2.\end{aligned}$$

Therefore, $|B^p_{G'\setminus E}(K)| \ge |B^1_{G'\setminus E}(K)| + \frac{2+3+\cdots+p}{2} \ge p^2/4$ and

$$|B^p_{G'\setminus E}(K)| \cdot \rho(|B^p_{G'\setminus E}(K)|) \ge p^2/4 \cdot \rho(p^2/4) \ge p+1,$$

where the last inequality follows since $p > p_0$. This completes the proof of (7). ◻

Lemma 4.4. *Let $0 < \varepsilon_1 \le 1/400, 0 < \varepsilon_2 < 1/2, \eta \ge \max\{\frac{40\varepsilon_1}{\log 3}, 5\varepsilon_2\}$, and $2 \le s \le t$. Then there exists d_0 such that the following holds for any $d \ge d_0$. Let G be an n-vertex $K_{s,t}$-free $(\varepsilon_1, \varepsilon_2 d)$-robust-expander with $d(G) = d < \log^{200s} n$ and $\Delta(G) \le d \log^{120} n$. Then G contains a clique immersion of order at least $\ell' = (1-4\eta)d$.*

Proof. Let $k = \log n/(800s \log\log n)$ and $r = (\log\log n)^5$. First, we claim that there are at least d vertices of degree at least $d(G) - 2\eta d$ which are pairwise a distance at least $2k+1$ far apart. Indeed, letting L be the set of vertices with degree at least $d(G) - 2\eta d$, as $\Delta(G) \le d \log^{120} n$, we see that $d(G)n < (d(G) - 2\eta d) \cdot (n - |L|) + d \log^{120} n \cdot |L|$, implying that $|L| > n/\log^{121} n$. Let Y be a maximal set of vertices in L which are pairwise a distance at least $2k+1$ apart. Suppose to the contrary $|Y| < d$. Since $d < \log^{200s} n$ and that $\Delta(G) \le d \log^{120} n$,

$$|B^{2k}(Y)| < 2d \cdot \Delta(G)^{2k} < n/\log^{121} n.$$

Thus there exists some vertex $v \in L \setminus B^{2k}(Y)$, contradicting the maximality of Y.

Let then $v_1, v_2, \cdots, v_{\ell'}$ be such vertices, which will serve as the branch vertices of the clique immersion. By choice, all the balls $B^k_G(v_i)$, $i \in [\ell']$, are pairwise vertex disjoint. Let $I \subseteq \binom{[\ell']}{2}$ be a maximal subset for which we can find paths $P_e, e \in I$, so that the following hold.

- **B1** For each $ij \in I$, P_{ij} is a (v_i, v_j)-path with length at most $2\log^4 n$.
- **B2** For distinct $e, e' \in I$, the paths P_e and $P_{e'}$ are edge disjoint.
- **B3** For each $e \in I$ and $i \notin e$, $E(B^r_G(v_i))$ and $E(P_e)$ are disjoint.
- **B4** For each $i \in [\ell']$, the subcollection P_e, $e \in I$ with $i \in e$, forms edge-disjoint consecutive shortest paths from v_i in $B^r_G(v_i)$.

If $I = \binom{[\ell']}{2}$, then by **B2**, we have a $K_{\ell'}$-immersion. Suppose there exists some $ij \in \binom{[\ell']}{2} \setminus I$. Let $W = \bigcup_{e \in I} E(P_e)$. By **B1** and **B2**, we have $|W| \le \binom{\ell'}{2} \cdot 2\log^4 n \le d^2 \log^4 n$. By **B4**, we can apply Lemma 4.3 with $t = r$, $D = d^2 \log^7 n$, $K = \{v_i\}$, $Z = \varnothing$ and $E = W$ to get $|B^r_{G \setminus W}(v_i)| \ge d^2 \log^7 n$. Furthermore, by Lemma 4.1 with $X = B^r_{G\setminus W}(v_i)$ and $Y = \bigcup_{e \in I} V(P_e)$, we have $|B^{k+r}_{G\setminus W}(v_i)| \ge |B^k_{G-Y}(X)| \ge \exp(\sqrt[4]{k}) \ge x := \exp(\sqrt[5]{\log n})$. Similarly $|B^{k+r}_{G\setminus W}(v_j)| \ge x$.

Let $W' = W \cup (\bigcup_{i' \ne i,j} E(B^r_G(v_{i'})))$, that is, the set of all edges that are either used in some connection or are in some inner ball of other branch vertices $v_{i'}$. As we chose the vertices v_i to be pairwise at least a distance $2k+1 > k+2r$ apart, none of the edges in W' is incident to $B^{k+r}_{G\setminus W}(v_i)$ or $B^{k+r}_{G\setminus W}(v_j)$. Note that $|W'| \le |W| + \Delta(G) \cdot 2(1-4\eta)d \cdot \Delta(G)^r \le d(G) \cdot \rho(x)x$. Therefore, by Lemma 2.4, there is a $(B^{k+r}_{G\setminus W}(v_i), B^{k+r}_{G\setminus W}(v_j))$-path in $G \setminus W'$ with length at most $m \le \log^4 n$. Thus, if we let P_{ij} be a shortest v_i, v_j-path in $G \setminus W'$, then P_{ij} has length at most $\log^4 n + 2k + 2r \le 2\log^4 n$. The paths $P_e, e \in I \cup \{ij\}$ satisfy the conditions **B1**–**B4** above, contradicting the maximality of I. □

4.2 Bounding size of Z_1

The following claim builds a large clique immersion with many vertices of relatively large degree. Recall that $Z_1 = \{v \in V(G) \mid d(v) \ge dm^3\}$.

Claim 4.5. If $|Z_1| \ge d$, then G contains a K_d-immersion.

Proof. Let $v_1, v_2, \cdots, v_d$ be distinct vertices in Z_1. We shall construct a K_d-immersion with all v_i as branch vertices. Let $I = \{\{i,j\} \in \binom{[d]}{2} \mid v_i v_j \notin E(G)\}$. Greedily connect v_i, v_j with $\{i,j\} \in I$ through their neighborhoods $N_i = N_G(v_i), N_j = N_G(v_j)$ using paths of length at most m. Let N_i and N_j be the current pair to connect. Then avoid using edges that are used in previous connections and all other v_p, $p \ne i,j$. Note that an (N_i, N_j)-path together with the corresponding incident edges forms a (v_i, v_j)-path of length at most $2m$.

The total number of edges used in all connections is at most $\binom{d}{2}\cdot 2m \le d^2 m$. It remains to show that we can connect all pairs in I as above. Note that each v_i has at least $dm^3 - d \ge dm^3/2$ incident edges which are not used in previous paths. Thus, each v_i can still reach at least $dm^3/2 := x$ neighbors in N_i. Recall that we avoid using at most $d^2 m \le d(G)\cdot\rho(x)\cdot x$ edges and at most $d \le \rho(x)\cdot x/4$ vertices. Hence, by Lemma 2.4, we can find a desired (N_i, N_j)-path of length at most m. □

Thus, it remains to consider the case $|Z_1| < d$. The following claim guarantees that the subgraph $G - Z_1$ still has large average degree, which allows us to restrict our attention to the subgraph $G - Z_1$.

Claim 4.6. $d(G - Z_1) \ge d(G) - \eta d$.

Proof. We may assume that $|Z_1| \ge \eta d$, otherwise we are done. It suffices to show that $e(G[Z_1]) + e(Z_1, V(G) - Z_1) \le \eta dn/2$. It is easy to see that $e(G[Z_1]) \le |Z_1|^2 = o(dn)$ because $d < \log^{200s} n$. By $K_{s,t}$-freeness, Lemma 2.1 implies that $e(Z_1, V(G)\setminus Z_1) = O(d^{1-1/s}n) = o(dn)$. □

4.3 Small expanders in $G - Z_1$

Let $G' = G - Z_1$. By Claim 4.6, $d(G') \ge d(G) - \eta d$. We now proceed to find small expanders of large average degree that are pairwise far apart from each other in G'. Recall that $k = \log n/(800s\log\log n)$. Let $\mathcal{F}$ be a maximal family of subgraphs in G' satisfying the following.

- **C1** Each $F \in \mathcal{F}$ is an $(\varepsilon_1, \varepsilon_2 d)$-robust-expander with $d(F) \ge (1 - 3\eta)d$.
- **C2** For distinct $F, F' \in \mathcal{F}$, $B^k_{G'}(V(F)) \cap B^k_{G'}(V(F')) = \varnothing$.

For each $F \in \mathcal{F}$, let $n_F = |F|$, $m_F = \frac{2}{\varepsilon_1}\log^3\left(\frac{15n_F}{\varepsilon_2 d}\right)$. If $d(F) \ge \log^{200s} n_F$ or $\Delta(F) \le d\log^{120} n_F$, then by the proof of dense case in Section 3, or by Lemma 4.4, we have a clique immersion of order at least $(1-5\eta)d(F) \ge (1-9\eta)d$, finishing the proof. Thus we may assume that for each $F \in \mathcal{F}$,

$$d(F) < \log^{200s} n_F, \quad \Delta(G') \ge \Delta(F) \ge d\log^{120} n_F.$$

Recall that $\Delta(G') \le dm^3 \le d\log^{12} n$, therefore

$$\exp(\sqrt[200s]{d/2}) \;\le\; n_F \;\le\; \exp(\sqrt[10]{\log n}). \tag{9}$$

We claim that $|\mathcal{F}| \ge d$. Otherwise, let $U = \bigcup_{F\in\mathcal{F}} B^{2k}_{G'}(V(F))$. Then

$$|U| \le |\mathcal{F}|\cdot 2\max_{F\in\mathcal{F}} n_F \cdot \Delta(G')^{2k} \le d\exp(\sqrt[10]{\log n})(dm^3)^{2k} < n/m^4$$

and it follows that $d(G' - U) \ge d(G') - 2|U|\cdot dm^3/n \ge (1-2\eta)d$. Lemma 2.3 implies that $G' - U$ contains an $(\varepsilon_1, \varepsilon_2 d)$-robust-expander F' with $d(F') \ge (1-3\eta)d$, which, by choice, is a distance at least $2k+1$ away from all expanders in $\mathcal{F}$, contradicting the maximality of $\mathcal{F}$.

4.4 Kernels in subexpanders

We have shown that $\mathcal{F}$ contains d expanders, say $F_1, F_2, \cdots, F_d$. Since each F_i satisfies $d(F_i) \ge (1-3\eta)d$, we can choose distinct vertices $v_1, v_2, \cdots, v_d$ as the branch vertices such that $v_i \in V(F_i)$ and $d_{F_i}(v_i) \ge (1-3\eta)d, i \in [d]$. Note that the graph G' may not have the expansion property as in (3) for small sets. Nonetheless, we shall see that each ball centered at v_i still expands well in the following two stages. We will firstly construct a collection of 'kernels' K_i containing v_i, which are sets of vertices in F_i that locally maintain certain robust expansion property. Then by further expanding (subsets of) the kernels in $G' = G - Z_1$, we can follow the strategy in the proof of Lemma 4.4 to finish the embedding of a desired clique immersion.

Note that for sufficiently large d, by (9), each small expander F_i has at least $\exp(\sqrt[200s]{d/2}) \ge d^2$ vertices. For each v_i, we shall show that, it expands robustly in F_i as follows.

D Let $P_1, P_2, \cdots, P_q$ be edge-disjoint consecutive shortest paths from v_i in F_i such that $q < \ell'$ and denote $E = \bigcup_{j\in[q]} E(P_j)$. Let s_i be the minimum integer such that $|B^{s_i}_{F_i\setminus E}(v_i)| \geq d^2$. Then $s_i \leq \log^4 d$.

To see this, note that as $d_{F_i}(v_i) \geq (1-3\eta)d \geq \ell' + \varepsilon_2 d$, applying Lemma 4.3 with $G = F_i$, $t = \log^4 d$, $D = d^2, K = \{v_i\}$ and $Z = \varnothing$, we have for large d that $|B^s_{F_i\setminus E}(v_i)| \geq d^2$.

For each $i \in [d]$, let $K_i = B^{s_i}_{F_i}(v_i)$. Using **D**, we see that large subsets of K_i expand robustly in G'. Recall that $r = (\log\log n)^5$. Note also that by **C2**, all balls $B^r_{G'}(K_i)$ are pairwise disjoint.

E Let $K' \subseteq K_i$ be a subset of size at least d^2 and $P_1, P_2, \cdots, P_q$ be edge-disjoint consecutive shortest paths from K' within $B^r_{G'}(K')$ in G' such that $q < \ell'$ and denote $E = \bigcup_{j\in[q]} E(P_j)$. Then $|B^r_{G'\setminus E}(K')| \geq d^2 \log^7 n$.

Indeed, such K' satisfies $|K'| \cdot \rho(|K'|)/4 \geq d > |Z_1|$ and then $N_{G'}(K') \geq d$. Therefore, **E** follows from Lemma 4.3 with $t = r, D = d^2 \log^7 n, K = K'$ and $Z = Z_1$.

4.5 Building clique immersion

We are now ready to finish the proof by iteratively finding edge disjoint paths connecting all pairs of branch vertices $v_1, v_2, \cdots, v_{\ell'}$. To do this, we will expand each v_i in three stages (to K_i', $B^r_{G'\setminus W}(K_i')$ and then $B^{k+r}_{G'\setminus W}(K_i')$ as depicted in Figure 2).

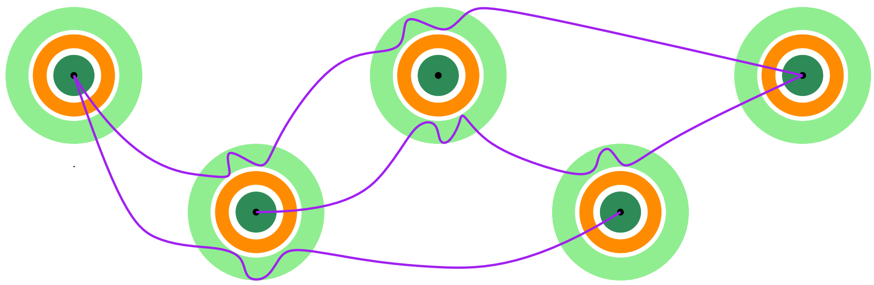

Figure 2: Final construction

By property **C2**, all the outer balls $B^k_{G'}(K_i)$ are pairwise disjoint, $i \in [\ell']$. Let $I \subset \binom{[\ell']}{2}$ be a maximal subset for which we can find pairwise edge-disjoint paths $P_e, e \in I$, so that for each $ij \in I$, P_{ij} is a (v_i, v_j)-path with length at most $2\log^4 n$ and

F1 for each $e \in I$ and $i \notin e$, $B^r_{G'}(K_i)$ and P_e are vertex disjoint;

F2 for each $i \in [\ell']$, the subcollection P_e, $e \in I$ with $i \in e$, form edge-disjoint consecutive shortest paths from v_i in $B^{s_i}_{F_i}(v_i)$, and $P_e - K_i$, $e \in I$ with $i \in e$, are edge-disjoint consecutive shortest paths from K_i in $B^r_{G'}(K_i)$.

We may assume there is $ij \in \binom{[\ell']}{2} \setminus I$ for otherwise P_e, $e \in I$, form a desired clique immersion. We will reach a contradiction by finding a (v_i, v_j)-path that is short and additionally avoids all the edges used in previous connections and all vertices in the inner balls of other branch vertices $v_p, p \neq i, j$.

Let $W = \bigcup_{e\in I} E(P_e)$ and $W' = \bigcup_{e\in I} V(P_e)$. Then $|W|, |W'| \leq d^2 \log^4 n$. By **D**, **F1** and **F2**, for $i \in [\ell']$, $|B^{s_i}_{F_i\setminus W}(v_i)| \geq d^2$. Let $K_i' = B^{s_i}_{F_i\setminus W}(v_i)$ and it follows from **F1**, **F2** and **E** that $|B^r_{G'\setminus W}(K_i')| \geq d^2 \log^7 n$.

Next, by Lemma 4.1 with $X = B^r_{G'\setminus W}(K_i'), Y = (W' \setminus \{v_i\}) \cup Z_1$, we have

$$|B^{k+r}_{G'\setminus W}(K_i')| \geq |B^k_{G-Y}(X)| \geq \exp(\sqrt[4]{k}) \geq x := \exp(\sqrt[5]{\log n}).$$

Let $W^* = \bigcup_{p \neq i,j} B^r_{G'}(K_p)$. As we choose the kernels K_i to be pairwise at least a distance $2k+1 > k+2r$ apart, both $B^{k+r}_{G' \setminus W}(K'_i)$ and $B^{k+r}_{G' \setminus W}(K'_j)$ are disjoint from W^*. Recall from 9 that $|K_i| \leq |F_i| \leq \exp(\sqrt[10]{\log n})$, for each $i \in [\ell']$ and $\Delta(G') \leq dm^3$, so the total number of vertices we avoid in **F1** is $|W^*| \leq (1-4\eta)d \cdot 2\exp(\sqrt[10]{\log n})(dm^3)^r \leq \rho(x)x/4$. Moreover, the number of edges we avoid is $|W| \leq d^2 \log^4 n \leq d(G)\rho(x)x$. Since $|B^{k+r}_{G' \setminus W}(K'_i)|$, $|B^{k+r}_{G' \setminus W}(K'_j)| \geq x$, by Lemma 2.4, there exists a $(B^{k+r}_{G' \setminus W}(K'_i), B^{k+r}_{G' \setminus W}(K'_j))$-path of length at most $\log^4 n$ in $(G-W^*) \setminus W$. Thus, if we let P_{ij} be a shortest (v_i, v_j)-path in $(G-W^*) \setminus W$, then P_{ij} has length at most $\log^4 n + 2k + 2r + 2\log^4 d \leq 2\log^4 n$. The paths $P_e, e \in I \cup \{ij\}$ satisfy **F1** and **F2**, contradicting the maximality of I.

This completes the proof.

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