Extremal graphs for blow-ups of cycles and trees

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Abstract

The *blow-up* of a graph H is the graph obtained from replacing each edge in H by a clique of the same size where the new vertices of the cliques are all different. Erdős et al. and Chen et al. determined the extremal number of blow-ups of stars. Glebov determined the extremal number and found all extremal graphs for blow-ups of paths. We determined the extremal number and found the extremal graphs for the blow-ups of cycles and a large class of trees, when n is sufficiently large. This generalizes their results. The additional aim of our note is to draw attention to a powerful tool, a classical decomposition theorem of Simonovits.

1 Introduction

Notation in this note is standard. We consider undirected graphs without loops and multi-edges. For a graph G, denote by $E(G)$ the set of edges and $V(G)$ the set of vertices of G. The *order* of a graph is the number of its vertices. The number of edges of G is denoted by $e(G) = |E(G)|$. For $U \subseteq V(G)$, let $G[U]$ be the subgraph of G induced by U. A path on k vertices is denoted by P_k, a star with $k+1$ vertices is denoted by S_k and a cycle with k edges is denoted by C_k. A *matching* in G is a set of vertex disjoint edges from $E(G)$, denote by M_k a matching of size k. Denote by $T_{n,p}$ the p-class *Turán graph*, namely the complete p-partite graph on n vertices with the size of each partite set as equal as possible.

The *extremal number*, $\mathrm{ex}(n, H)$, of a graph H is the maximum number of edges in a graph on n vertices which does not contain H as a subgraph. An H-free n-vertex graph with $\mathrm{ex}(n, H)$ edges is called an *extremal graph* for H, or H*-extremal*. Turán [13, 14] showed that $T_{n,p}$ is the unique extremal graph for K_{p+1}. The Erdős-Stone-Simonovits Theorem [4, 6] states that asymptotically Turán's construction is best-possible for any $(p+1)$-chromatic graph H (as long as $p \geq 2$). More precisely $\mathrm{ex}(n, H) = \left(1 - \frac{1}{p}\right)\frac{n^2}{2} + o(n^2)$.

Given a graph H, the *blow-up* of H, denoted as H^{p+1}, is the graph obtained from replacing each edge in H by a clique of size $p+1$ where the new vertices of the cliques are all different (see Figure 1(a)).

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Erdős, Füredi, Gould and Gunderson [3] determined, for sufficiently large n, the extremal number for triangles intersecting in exactly one common vertex. One can think of this graph as blowing up edges of a star to triangles. More generally, k cliques of size $p+1$ intersecting in exactly one common vertex is S_k^{p+1}, Chen, Gould, Pfender and Wei [2] generalized the main result of [3] to S_k^{p+1}:

Theorem 1. *[2] For any $p \geq 2$ and $k \geq 1$, and for any $n \geq 16k^3(p+1)^8$, we have*

$$\mathrm{ex}(n, S_k^{p+1}) = \mathrm{ex}(n, K_{p+1}) + \begin{cases} k^2 - k & \text{if } k \text{ is odd,} \\ k^2 - \frac{3}{2}k & \text{if } k \text{ is even.} \end{cases}$$

Given two vertex-disjoint graphs H and G, denote by $H \bigotimes G$ the graph obtained by joining each vertex of H to each vertex of G. Let $H(n,p,s)$ be $K_{s-1} \bigotimes T_{n-s+1,p}$ (see Figure 1(b)) and $H'(n,p,s)$ be any of the graphs obtained by putting one extra edge in any class of $T_{n-s+1,p}$ in $H(n,p,s)$.

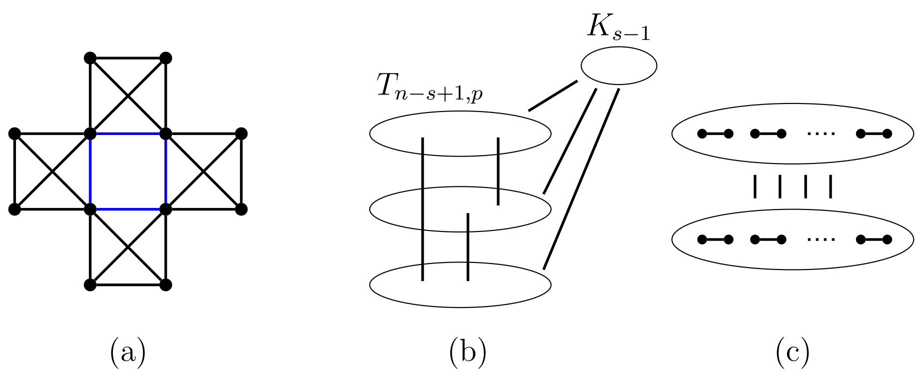

Figure 1: (a) C_4^4; (b) $H(n,p,s)$; (c) $H^*(n)$.

Recently, Glebov [7] determined, for sufficiently large n, the extremal number and the extremal graphs for the blow-up of paths. More history of this topic are given in Section 2.

Theorem 2. *[7] For any $p \geq 2$ and $k \geq 1$, and for any $n > 16k^{11}(p+1)^8$, $H(n,p,\lfloor\frac{k-1}{2}\rfloor+1)$ ($H'(n,p,\lfloor\frac{k-1}{2}\rfloor+1)$ resp.) is the unique extremal graph for P_{k+1}^{p+1} when k is odd (even resp.).*

The main motivation for this note is that [2], [3] and [7] give sporadic results about problems of the same flavor. We unite these extremal problems for blow-ups of graphs and look at the general theory behind these results by investigating the decomposition families of the forbidden graphs. Using the method in [10] (see also [9], [11]), we determine the extremal number and found all extremal graphs for all blow-ups of cycles. Somewhat surprisingly, the result for blow-ups of cycles is not much different from blow-ups of paths except for C_3^3. Before stating our results, we need a definition. Let $H^*(n)$ be graphs obtained by putting (almost) perfect matchings in both classes in $K_{\lceil n/2\rceil,\lfloor n/2\rfloor}$ (see Figure 1(c)).

Theorem 3. *For any $p \geq 2$ and $k \geq 3$, when n is sufficiently large, we have the following results:*

(i) For any $C_k^{p+1} \neq C_3^3$, $H(n, p, \lfloor \frac{k-1}{2} \rfloor + 1)$ ($H'(n, p, \lfloor \frac{k-1}{2} \rfloor + 1)$ resp.) is the unique extremal graph for C_k^{p+1} when k is odd (even resp.).

(ii) For C_3^3, if $4|n$, $H^*(n)$ is the unique extremal graph; otherwise both $H^*(n)$ and $H(n, 2, 2)$ are extremal graphs.

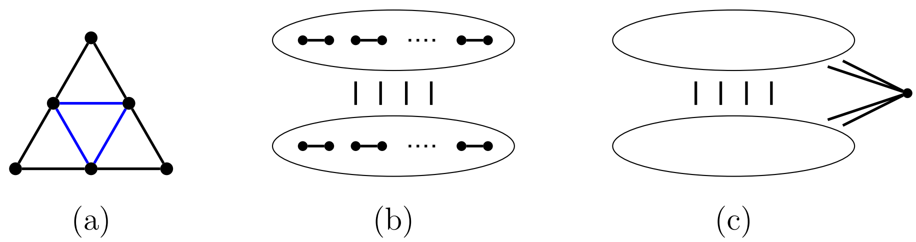

Figure 2: (a) C_3^3; (b) $H^*(n)$; (c) $H(n, 2, 2)$.

Remark 4. Both $H'(n, p, s)$ and $H^*(n)$ might contain non-isomorphic graphs. However graphs in the same family are similar in the sense that they have the same number of edges. Since their difference does not matter in this note, we will treat each of them as a "unique" graph instead of families of graphs.

In addition, for a large class of trees, we determined the extremal number for their blow-up graphs and found their unique extremal graph.

Theorem 5. *Given a tree T, denote by A and B its two color classes with $|A| \leq |B|$. For any $p \geq 3$, when n is sufficiently large, we have that*

(i) if T has a leaf in A and $\alpha(T) = |B|$, then $H(n, p, |A|)$ is the unique extremal graph for T^{p+1}.

(ii) if the minimum degree in A is 2, then $H'(n, p, |A|)$ is the unique extremal graph for T^{p+1}.

Remark 6. Trees considered in Theorem 5 (i) include even paths and those in (ii) include odd path. Hence it implies Theorem 2 when $p \geq 3$. For $p = 2$, the technique in the proof of Theorem 3 (see Appendix) works for blow-ups of paths. It is not difficult to see that in a proper subdivision of any star, its smaller color class either has minimum degree 2 or has a leaf and its independence number equals to the size of the larger color class, thus Theorem 5 can be applied to blow-ups of a proper subdivision of stars, which is an extention of Theorem 1.

The rest of this paper is organized as follows: in Section 2 we provide more motivation and the key lemma. Section 3 gives a proof for Theorem 3 when $p \geq 3$ and Section 4 is devoted to proof of Theorem 5 (i). The proof for Theorem 5 (ii) is similar, we include a sketch of its proof in Appendix together with a proof for Theorem 3 when $p = 2$.

We finish this section with a few more definitions that will be used later. We denote the *degree* of a vertex v by $d(v)$ and write $N(v)$ for the set of its neighbors and for $S \subseteq V(G)$, let $N(S)$ be the set of vertices that have some neighbors in S. Denote by K_t^- the graph obtained from deleting an edge from a complete graph on t vertices. A *dominating vertex* in G is a vertex that is adjacent to all other vertices in G. A *linear forest* is a forest whose connected components are paths. Two disjoint vertex sets U and W are *completely joined* in G if $uw \in E(G)$ for all $u \in U, w \in W$. Write kH for the vertex disjoint union of k copies of H. For two vertex-disjoint graphs H and G, denote by $H \cup G$ the disjoint union of H and G.

2 Motivation and History

Given a graph H, a *vertex split* on some vertex $v \in V(H)$ is defined as follows: replace v by an independent set of size $d(v)$ in which each vertex is adjacent to exactly one distinct vertex in $N_H(v)$. Given a vertex subset $U \subseteq V(H)$, a vertex split on U means applying vertex split on the vertices in U one by one. It is not difficult to see that the order of vertices we apply vertex split does not matter. Denote by $\mathcal{H}(H)$ the family of graphs that can be obtained from H by applying vertex split on some $U \subseteq V(H)$. Note that U could be empty, therefore $H \in \mathcal{H}(H)$. For example, $\mathcal{H}(P_{k+1})$ is the family of all linear forests with k edges and $\mathcal{H}(C_k)$ consists of C_k and all linear forests with k edges. Given a family $\mathcal{L}$, define $p = p(\mathcal{L}) = \min_{L \in \mathcal{L}} \chi(L) - 1$.

Definition 7. Denote by I_v the v-vertex graph with no edges. Given a family $\mathcal{L}$, let $\mathcal{M} := \mathcal{M}(\mathcal{L})$ be the family of minimal graphs M for which there exist an $L \in \mathcal{L}$ and a $t = t(L)$ such that $L \subseteq M' \bigotimes K_{p-1}(t, \dots, t)$, where $M' = M \cup I_t$. We call $\mathcal{M}$ the *decomposition family* of $\mathcal{L}$.

Thus, a graph M is in $\mathcal{M}$ if the graph obtained from putting M into a class of a large $T_{n,p}$ contains some $L \in \mathcal{L}$. If $L \in \mathcal{L}$ with minimum chromatic number $p+1$, then $L \subseteq K_{p+1}(t, \dots, t)$ for some $t \geq 1$, therefore the decomposition family $\mathcal{M}$ always contains some bipartite graphs.

Example 8. Denote by O_6 the edge-graph of the octahedron, namely $O_6 = K_{2,2,2}$. Since $O_6 = C_4 \bigotimes I_2$, we have that $\mathcal{M}(O_6) = \{C_4\}$. For any $\ell > 1$, $\mathcal{M}(C_{2\ell+1}) = \{P_2\}$. For blow-ups of stars, $\mathcal{M}(S_k^{p+1}) = \{S_k, M_k\}$ for $p \geq 2$ and $k \geq 1$; for blow-ups of paths $\mathcal{M}(P_{k+1}^{p+1}) = \{$all linear forests with k edges$\} = \mathcal{H}(P_{k+1})$ and for cycles $\mathcal{M}(C_k^{p+1}) = \{C_k$, all linear forests with k edges$\} = \mathcal{H}(C_k)$ for $p \geq 3$ and $k \geq 1$.

For a family of forbidden graph $\mathcal{L}$ with decomposition family $\mathcal{M}$, we have

$$e(T_{n,p}) + \mathrm{ex}(\frac{n}{p}, \mathcal{M}) \leq \mathrm{ex}(n, \mathcal{L}) \leq e(T_{n,p}) + (1 + o(1))p \cdot \mathrm{ex}(\frac{n}{p}, \mathcal{M}), \tag{1}$$

where the lower bound is obtained from putting an $\mathcal{M}$-extremal graph in one of the classes in $T_{n,p}$ (see [1]).

The Erdős-Stone-Simonovits Theorem determines asymptotically the extremal functions of non-bipartite graphs, while the decomposition family governs the finer error terms as shown in (1), thus it helps to give sharper bounds on the extremal number.

There are examples where the upper bound in (1) holds. Let $Q(r,p)$ be the graph consisting of a dominating vertex and a p-class Turán graph on rp vertices, namely $Q(r,p) = T_{rp,p} \bigotimes I_1$. Notice that $\mathcal{M}(Q(r,p)) = \{S_r\}$, thus an $\mathcal{M}(Q(r,p))$-free graph has maximum degree $r-1$. Simonovits [8] showed that a $Q(r,p)$-extremal graph can be obtained from putting (almost) $(r-1)$-regular triangle-free graphs into each class of a $T_{n,p}$.

We recall the Octahedron Theorem by Erdős and Simonovits [5], which gives an example where neither the upper bound nor the lower bound in (1) is true.

Theorem 9. *For sufficiently large n, every O_6-extremal graph S_n can be obtained as $S_n = U_m \bigotimes Z_{n-m}$, for some m-vertex C_4-extremal graph U_m and some $(n-m)$-vertex P_3-extremal graph Z_{n-m}, where $m = n/2 + o(n)$.*

A graph L is *weakly edge-color-critical*, or shortly *weakly-critical*, if there is an edge $e \in E(L)$ for which $\chi(L-e) < \chi(L)$. Simonovits [8] proved that the Turán graph is the unique extremal graph for weakly-critical graphs when n is sufficiently large. In the same paper, he also proved when the forbidden graph is $L = sH$, where H is weakly-critical, and $\chi(H) = p+1 \geq 3$, then for sufficiently large n, the unique L-extremal graph is $H(n,p,s)$. Later in [10], he further generalized this result to the following theorem.

Theorem 10. *Let $\mathcal{L}$ be the family of forbidden graphs and $p = p(\mathcal{L}) = \min_{L\in\mathcal{L}} \chi(L) - 1$. If by omitting any $s-1$ vertices of any $L \in \mathcal{L}$ we obtain a graph with chromatic number at least $p+1$, but by omitting s suitable edges of some $L \in \mathcal{L}$ we get a p-colorable graph, then $H(n,p,s)$ is the unique extremal graph for n sufficiently large.*

Simonovits [12] asked the following question.

Question 11. *Characterize graphs whose unique extremal graph is of the form $H(n,p,s)$.*

We make a step towards answering Question 11: notice that the blow-ups of cycles and trees does not satisfy the hypothesis in Theorem 10, hence Theorem 3 and 5 provides an additional family of forbidden graphs whose unique extremal graph is $H(n,p,s)$ for suitable p and s.

On the other hand, the results in [2], [3], [7], [8] and [10] show that for blow-ups of stars, paths and many other families of graphs, the lower bound construction is optimal. Our results show that for blow-ups of cycles and a large class of trees, this is also the case. It would be interesting to describe all decomposition families $\mathcal{M}$ where the lower bound in (1) is sharp. Here we make the first attempt towards this direction.

The following lemma shows that the decomposition family of blow-ups of some graphs (in particular bipartite graphs) is actually the family obtained from splitting its vertices.

Lemma 12. *Given $p \geq 3$ and any graph H with $\chi(H) \leq p-1$, $\mathcal{M}(H^{p+1}) = \mathcal{H}(H)$. In particular, a matching of size $e(H)$ is in $\mathcal{M}(H^{p+1})$.*

Proof. Note that $\chi(H^{p+1}) = p+1$, by the definition of decomposition family, any graph M in $\mathcal{M}(H^{p+1})$ is a minimal graph under the condition that there exists a copy of M in H, such that removing the vertex set of M together with a suitable independent set I_v results in a $(p-1)$-colorable graph. Since I_v is an independent set, it can have at most one vertex from each $(p+1)$-clique in H^{p+1}. Recall the removal of $M \cup I_v$ decreases the chromatic number by at least two, this implies M should have at least one vertex from each $(p+1)$-clique in H^{p+1}. Since a vertex from M and a vertex from I_v in the same $(p+1)$-clique would be adjacent, which contradicts the minimality of M, thus I_v is empty and M includes exactly two vertices from each $(p+1)$-clique in H^{p+1}. Locally, for each $v \in H$, $N_H(v) \cup \{v\}$ spans a $S^{p+1}_{d_H(v)}$ in H^{p+1}. If the center of that $S^{p+1}_{d_H(v)}$ is in M, then M also contains exactly one other vertex from each of the $(p+1)$-cliques in this $S^{p+1}_{d_H(v)}$. This implies v is not a split vertex in M. Otherwise, M contains two vertices from each $(p+1)$-cliques of $S^{p+1}_{d_H(v)}$, which implies that v was split into $d_H(v)$ leaves in M. Hence $\mathcal{M}(H^{p+1}) = \mathcal{H}(H)$. In particular, by splitting all vertices of H, we obtain a matching of size $e(H)$. $\square$

The following definition was introduced in [10].

Definition 13. Denote by $\mathbb{D}(n, p, r)$ the family of n-vertex graphs G_n satisfying the following symmetry condition:

(i) It is possible to omit at most r vertices of G_n so that the remaining graph G' is a product of graphs of almost equal order: $G' = \prod_{i \leq p} G^i$, where $|V(G^i)| = n_i$ and $|n_i - \frac{n}{p}| \leq r$, for every $i \leq p$.

(ii) For every $i \leq p$, there exist connected graphs H_i such that $G_i = k_i H_i$, where $k_i = \frac{n_i}{|V(H_i)|}$ and any two copies H_i^j, H_i^ℓ in G_i $(1 \leq j < \ell \leq k_i)$, are *symmetric subgraphs* of G_n: there exists an isomorphism $\omega : H_i^j \to H_i^\ell$ such that for every $x \in H_i^j$, $u \in G_n - H_i^j - H_i^\ell$, $xu \in E(G_n)$ if and only if $\omega(x)u \in E(G_n)$.

The graphs H_i will be called the *blocks*, the vertices in $G_n - G'$ will be called *exceptional* vertices (see Figure 3).

We will need the following two results of Simonovits ([10, 11]).

Theorem 14. *[10] Assume that a finite family $\mathcal{L}$ of forbidden graphs with $p(\mathcal{L}) = p$ is given. If for some $L \in \mathcal{L}$ and $\ell := |V(L)|$,*

$$L \subseteq P_\ell \bigotimes K_{p-1}(\ell, \ell, \ldots, \ell), \tag{2}$$

then there exist $r = r(L)$ and $n_0 = n_0(r)$ such that $\mathbb{D}(n, p, r)$ contains an $\mathcal{L}$-extremal graph for every $n > n_0$. Furthermore, if this is the only extremal graph in $\mathbb{D}(n, p, r)$, then it is the unique extremal graph for every sufficiently large n.

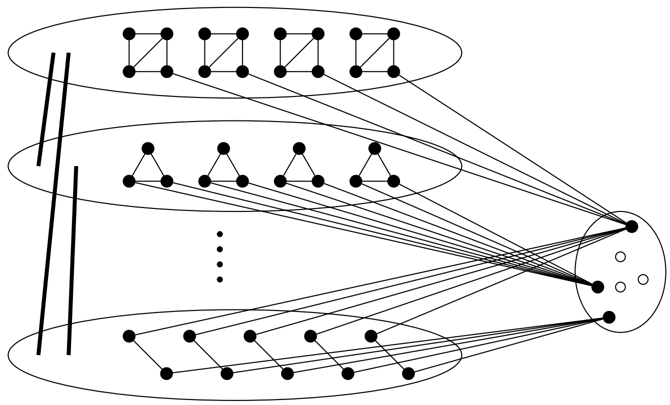

Figure 3: Symmetric subgraphs: blocks here are K_4^-, K_3 and P_2 respectively.

Theorem 15. *[11] Assume that a finite family $\mathcal{L}$ of forbidden graphs with $p(\mathcal{L}) = p$ is given. If for some $L \in \mathcal{L}$ and $\ell := |V(L)|$,*

$$L \subseteq \ell P_2 \bigotimes K_{p-1}(2\ell, 2\ell, \ldots, 2\ell), \tag{3}$$

then there exist $r = r(L)$ and $n_0 = n_0(r)$ such that $\mathbb{D}(n, p, r)$ contains an $\mathcal{L}$-extremal graph for every $n > n_0$. Furthermore for any $\mathcal{L}$-extremal graph $G \in \mathbb{D}(n, p, r)$, we have that

(i) all blocks of G will consist of isolated vertices: the product graph G' will be a Turán graph $T_{n',p}$.

(ii) each exceptional vertex in $G - G'$ is joined either to all the vertices of G' or to all the vertices of $p - 1$ classes of G' and to no vertex of the remaining class.

The key idea of the proof of Theorem 3 is using Theorem 15 to get a good vertex partition of an extremal graph of C_k^{p+1}. Then show that in this partition, there are t exceptional vertices, where $t = \lfloor \frac{k-1}{2} \rfloor$, and the remaining of the graph is a Turán graph (with one extra edge if k is even). This together with G being extremal imply $G \simeq H(n, p, t + 1)$ ($H'(n, p, t + 1)$ if k is even).

Remark 16. Assume first k is odd. Notice that $H(n, p, t + 1)$ is C_k^{p+1}-free. Indeed the set of t dominating vertices in $H(n, p, t + 1)$ together with one class of $T_{n-t,p}$ is $\mathcal{H}(C_k)$-free. By Lemma 12, $\mathcal{M}(C_k^{p+1}) = \mathcal{H}(C_k)$ when $p \geq 3$, and observe that when $p = 2$, $\mathcal{M}(C_k^3) \subseteq \mathcal{M}(C_k^{p+1})$. Thus $H(n, p, t + 1)$ is the product of $p - 1$ empty graphs and one $\mathcal{M}(C_k^{p+1})$-free graph, which is the lower bound construction in (1). Hence $H(n, p, t + 1)$ is C_k^{p+1}-free. Similarly when k is even, $H'(n, p, t + 1)$ is C_k^{p+1}-free.

3 Proof of Theorem 3

Proof. We prove Theorem 3 for $p \geq 3$. A crucial observation is that (2) is equivalent to $\mathcal{M}(C_k^{p+1})$ containing some linear forest of size at most ℓ and (3) is equivalent to $\mathcal{M}(C_k^{p+1})$

containing a matching of size at most ℓ. Since $P_{k+1} \in \mathcal{M}(C_k^{p+1})$, (2) is satisfied with $L = C_k^{p+1}, \ell = k+1$. Thus by Theorem 14, there exists an extremal graph G in $\mathbb{D}(n,p,r)$ of C_k^{p+1} for some r. It suffices to prove that $G \simeq H(n,p,t+1)$ ($H'(n,p,t+1)$ resp.) when k is odd (even resp.). Then Theorem 14 implies it would be the unique extremal graph. Since $M_k \in \mathcal{M}(C_k^{p+1})$, (3) is also satisfied with $L = C_k^{p+1}, \ell = k$. We can apply Theorem 15 to get a vertex partition of G. Let $A_1, \ldots, A_p$ be the p classes in $T_{n',p}$. Let W be the set of vertices in $G - T_{n',p}$ that are joined to all vertices in $T_{n',p}$ and let B_i be the set of vertices in $G - T_{n',p} - W$ that are joined to all the vertices in $T_{n',p}$ but A_i (see Figure 4(a)). Define $C_i = A_i \cup B_i$, for all i. Note that in G all the cross-edges between A_i and C_j with $i \neq j$ are present, there might be some missing edges between some B_i and B_j. Let $D_i \subseteq C_i$ consist of vertices with no neighbor in W. Recall that $\mathcal{M}(C_k^{p+1}) = \{C_k$, all linear forests with k edges$\}$. We will frequently use the following fact.

Claim 17. *For any $i \leq p$, $G[W \cup C_i]$ is $\mathcal{M}(C_k^{p+1})$-free.*

Proof. Notice that $W \cup C_i$ is completely joined to $\bigcup_{j \neq i} A_j$. If for some $M \in \mathcal{M}(C_k^{p+1})$, $M \subseteq G[W \cup C_i]$, then $M' \bigotimes K_{p-1}(n'/p, \ldots, n'/p) \subseteq G$. This, by the definition of decomposition family, implies $C_k^{p+1} \subseteq G$, a contradiction. □

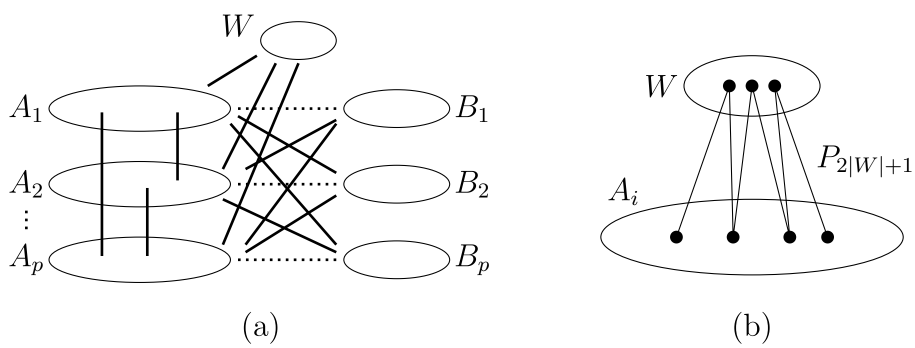

Figure 4: (a): a partition of G; (b): If $|W| > t$, then $2|W| + 1 \geq k+1$, $P_{k+1} \subseteq G[W \cup A_i]$.

Claim 18. $|W| = t = \lfloor \frac{k-1}{2} \rfloor$.

Proof. First of all, $|W| \leq t$. Indeed, by Claim 17, $G[W \cup C_i]$ is $\mathcal{M}(C_k^{p+1})$-free. Since W and A_i are completely joined, if $|W| > t$, then $P_{k+1} \subseteq G[W \cup A_i]$ (see Figure 4(b)). But $P_{k+1} \in \mathcal{M}(C_k^{p+1})$, a contradiction.

On the other hand, suppose $|W| \leq t-1$, then some simple calculation shows

$$e(G) \leq e(T_{n,p}) + \frac{t-1}{p} n + o(n).$$

However since $C_k^{p+1} \nsubseteq H(n,p,t+1)$, we have

$$e(G) \geq e(H(n,p,t+1)) \geq e(T_{n,p}) + \frac{t}{p}n + o(n),$$

a contradiction. $\square$

Case 1: k is odd. Then $t = \frac{k-1}{2}$. We shall show that for each $i \leq p$, $G[C_i]$ has no edge. Then by Claim 18 and the maximality of G, $G \simeq H(n,p,t+1)$. Indeed, any edge $xy \in C_i$ together with a $P_{2|W|+1} = P_k$ in $G[W \cup A_i]$ avoiding $\{x,y\}$ form a linear forest with k edges, which is in $\mathcal{M}(C_k)$. This contradicts Claim 17.

Case 2: k is even. Then $t = k/2 - 1$. It suffices to prove following claim. This together with Claim 18 and the maximality of G implies $G \simeq H'(n,p,t+1)$.

Claim 19. *When k is even, we have (i) each $G[C_i]$ has at most one edge; (ii) the edge in C_i has at least one endpoint not in D_i; and (iii) there is at most one class $G[C_i]$ having one such edge.*

Proof. (i): For contradiction suppose there are two edges in some $G[C_i]$, say e_1 and e_2. Then one can find a copy of $P_{2|W|+1} = P_{k-1}$ in $G[W \cup A_i]$ avoiding the endpoints of e_1, e_2, so we get a linear forest with k edges in $G[W \cup C_i]$, a contradiction.

(ii): Suppose $u_iv_i \in E(G[C_i])$ and $u_i, v_i \in D_i$. Since k is even hence $k \geq 4$, therefore W is not empty: $|W| = t = k/2 - 1 \geq 1$. Denote G_0 the graph obtained by deleting the edge u_iv_i and adding all edges between $\{u_i, v_i\}$ and W. There are at least two such cross-edges since W is nonempty and $u_i, v_i \in D_i$. Thus $e(G_0) > e(G)$. It remains to show G_0 is also C_k^{p+1}-free, which contradicts to the extremality of G. Notice that u_i and v_i is not joined to any vertex in C_i in G_0. Hence they have the same neighborhood as vertices in A_i. Also since G and G_0 only differ at u_i and v_i, a copy of C_k^{p+1} in G_0 must involve u_i or v_i or both. But then one can obtain a copy of C_k^{p+1} in G by replacing vertices from $\{u_i, v_i\}$ by vertices from A_i, a contradiction.

(iii): Suppose for some $i \neq j$, $u_iv_i \in E(G[C_i])$ and $u_jv_j \in E(G[C_j])$ with $u_i \notin D_i$ and $u_j \notin D_j$. Then we can find a copy of P_k in $G[C_i \cup W]$ which starts at a vertex in A_i and whose last edge is u_iv_i. Denote vertices of such a path $x_1, x_2, \ldots, x_{k-2}, u_i, v_i$ with $x_1 \in A_i$. We can then extend this path to a P_k^{p+1} in G (see Figure 5(a)).

Since $p \geq 3$, there is a third class, say A_ℓ with $\ell \neq i, j$. The last clique on this copy of P_k^{p+1}, namely the one containing u_i and v_i, intersect A_ℓ at exactly one vertex, call it u_ℓ. Then this P_k^{p+1} together with the $(p+1)$-clique consisting of one vertex from each A_q, $q \neq i, j, \ell$, and u_ℓ, x_1, u_j, v_j, form a C_k^{p+1} (see Figure 5(b)), where $x_1, \ldots, x_{k-2}, u_i, u_\ell, x_1$ is the vertices of C_k that was blown up. This yields a contradiction. $\square$

4 Proof of Theorem 5

In this section, unless otherwise specified, $p \geq 3$ and T is a tree with two color classes (partite sets) A and B such that $|A| \leq |B|$. Let $a = |A|$, $b = |B|$. Recall that, by Lemma 12, $\mathcal{M}(T^{p+1}) = \mathcal{H}(T)$. In particular, $T \in \mathcal{M}(T^{p+1})$.

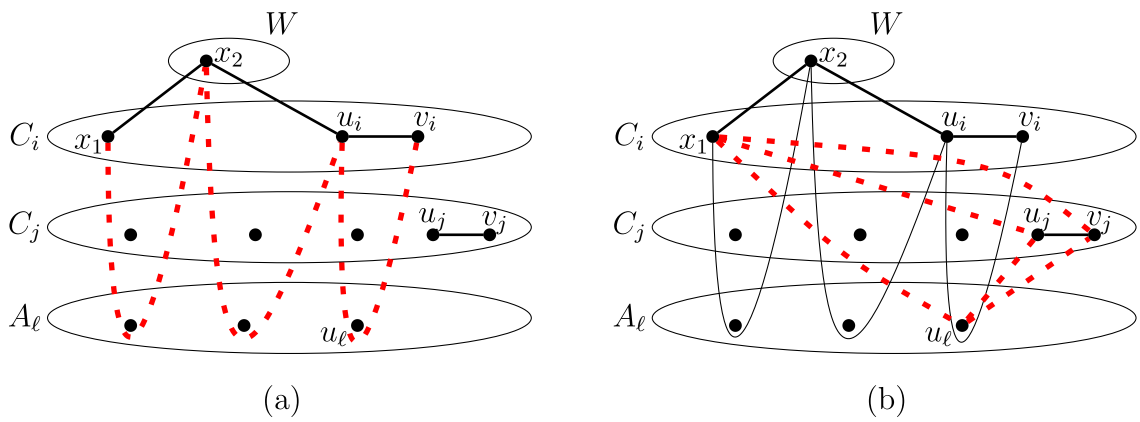

Figure 5: When $k = 4$: (a) Extend a P_k to a P_k^{p+1}; (b) Obtain a C_k^{p+1} from P_k^{p+1}.

Lemma 20. *If T has a leaf in A and $\alpha(T) = b$, then for any $m \geq 1$, $K_{a-1} \bigotimes \overline{K}_m$ is $\mathcal{H}(T)$-free, hence $\mathcal{M}(T^{p+1})$-free.*

Proof. For simplicity, let $G = K_{a-1} \bigotimes \overline{K}_m$ with $V(G) = X \cup Y$, where X is the set of vertices in the $(a-1)$-clique and Y is the remaining independent set. We may assume $|Y| = m \geq b+1$, otherwise $|V(G)| < |V(F)|$ for any $F \in \mathcal{H}(T)$. First notice that $T \not\subseteq G$. Indeed, an embedding of T in G has at least $|V(T)| - (a-1) = b+1$ vertices in Y, which contradicts to $\alpha(T) = b$. For contradiction, suppose that some forest $F \in \mathcal{H}(T)$ is in G. Recall that F is obtained from splitting vertices in some $U \subseteq V(T)$ to sets of leaves in F. For any v in U, denote by $L(v) \subseteq V(G)$ the set of leaves in F corresponding to v. We shall get a copy of T in G from a copy of F by applying the following operation to every $v \in U$ to undo the vertex split: pick any $v \in U$, look at the corresponding $L(v)$ in F. First discard edges in F adjacent to $L(v)$. Then if $L(v) \subseteq Y$, add in F edges from $N_G(L(v))$ to a vertex in Y; otherwise add in F edges from $N(L(v))$ to a vertex in $L(v) \cap X$ (see Figure 6). $\square$

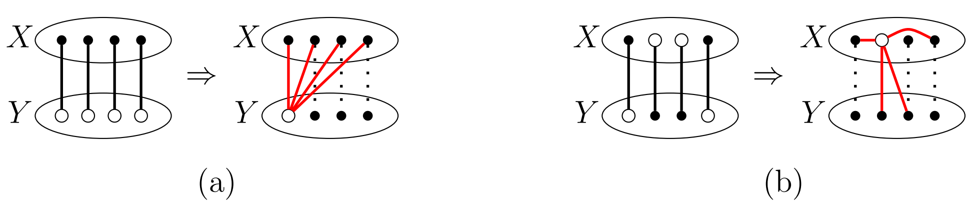

Figure 6: Circled vertices are in $L(v)$. (a): $L(v) \subseteq Y$; (b): $L(v) \cap X \neq \emptyset$.

From Lemma 20, we immediately get the following.

Proposition 21. *$H(n, p, a)$ is T^{p+1}-free.*

Proof. Since $H(n,p,a)$ is the product of $p-1$ independent sets and one $\mathcal{M}(T^{p+1})$-free graph (obtained from combining the set of $a-1$ dominating vertices with the remaining independent set). This is the lower bound construction in (1). Hence $H(n,p,a)$ is T^{p+1}-free. $\square$

Proof of Theorem 5 (i). Since a matching of size $e(T)$ is in $\mathcal{M}(T^{p+1})$, we can proceed as in the proof of Theorem 3 and define W, A_i, B_i, C_i in the same way. If $|W| \geq a$, then $T \subseteq G[W \cup C_i]$ for any i. However $T \in \mathcal{M}(T^{p+1})$, this implies $T^{p+1} \subseteq G$, a contradiction. If $|W| \leq a-2$, then

$$e(G) = e(T_{n,p}) + \frac{a-2}{p}n + o(n) < e(T_{n,p}) + \frac{a-1}{p}n + o(n) = e(H(n,p,a)),$$

a contradiction. Thus $|W| = a-1$. Also we may assume that W is non-empty. Indeed, if $W = \emptyset$, then $a = 1$. Since T has a leaf in A, it implies T is P_2, then T^{p+1} is K_{p+1} and its unique extremal graph is $H(n,p,1) = T_{n,p}$.

It remains to show that $e(G[C_i]) = 0$ for all $1 \leq i \leq p$. Let u be a leaf of T in A, and v be its neighbor in B. Let $T' = T - u$ and let F and F' be the forests obtained from splitting v in T and T' respectively. Notice that $F = F' \cup K_2$. Indeed, v has one more neighbor (leaf u) in T, which becomes a K_2 after splitting. Suppose there is an edge xy in $G[C_i]$ for some i. Since $u \in A$ and $|W| = a-1$, T' has an embedding in $G[W \cup C_i - \{x,y\}]$ with v in C_i. Splitting v in this copy of T' (not using x or y), we get a copy of F' in $W \cup C_i - \{x,y\}$. Note that for any $i \leq p$, $G[W \cup C_i]$ is $\mathcal{M}(T^{p+1})$-free, since $W \cup C_i$ is completely joined to A_j, $\forall j \neq i$. Thus edge xy together with this F' yields a copy of F in $G[W \cup C_i]$. However $F \in \mathcal{H}(T) = \mathcal{M}(T^{p+1})$, a contradiction. $\square$

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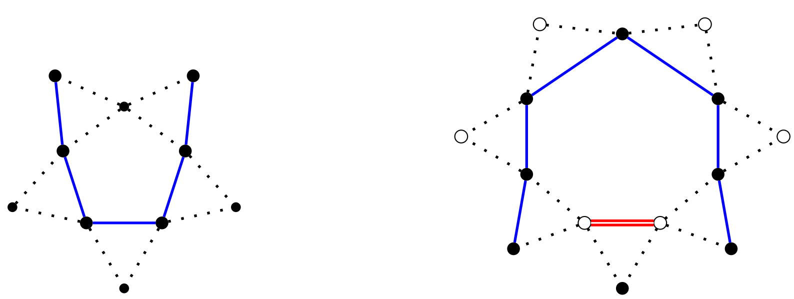
Appendix

Proof of Theorem 3 for $p = 2$. Given C_k^3, notice that $P_{k+1} \in \mathcal{M}(C_k^3)$ (See Figure 7(a)). Thus by Theorem 14, there is an extremal graph $G \in \mathbb{D}(n, 2, r)$ for C_k^3 and removing a few exceptional vertices from G results in $G' = A_1 \bigotimes A_2$, where A_1 and A_2 are disjoint unions of symmetric subgraphs (blocks) H_1 and H_2 respectively.



(a) (b)

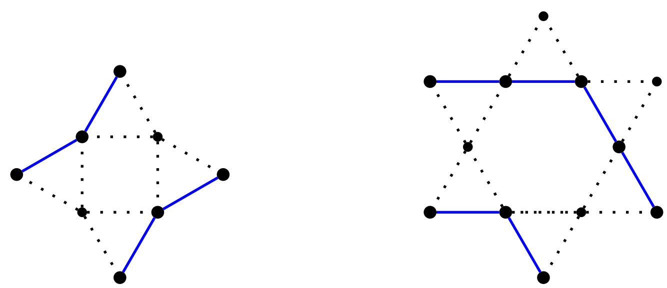
Figure 7: (a): $P_{k+1} \in \mathcal{M}(C_k^3)$ when $k = 5$; (b): $C_k^3 \subseteq (P_k \cup I_k) \bigotimes (P_2 \cup I_k)$ when $k = 7$.

Claim 22. *For any $k \geq 4$, each block H_i, $i = 1, 2$, is an isolated vertex. For C_3^3, H_i can be an isolated vertex or P_2, and $H^*(n)$ is the unique extremal graph when $4|n$, otherwise $H(n, 2, 2)$ is also an extremal graph.*

We first show how Claim 22 implies Theorem 3 for $p = 2$. When $k \geq 4$, since H_1 and H_2 are symmetric with respect to G, Claim 22 implies the exceptional vertices are adjacent either to all the vertices in G' or to all the vertices in one class of G' and none of the other class. Similarly let W be the set of vertices adjacent to all vertices of G', and B_1, B_2 be the sets of vertices joining only vertices in A_2 and A_1 respectively. Let $C_i = A_i \cup B_i$, for $i = 1, 2$. Claim 18 is still true, namely $|W| = t$.

When k is odd, $t = \frac{k-1}{2}$. It suffices to show $e(C_i) = 0$ for $i = 1, 2$. Indeed, the 2-coloring in Figure 7(b) shows that $C_k^3 \subseteq (P_k \cup I_k) \bigotimes (P_2 \cup I_k)$. Suppose C_i has an edge, since $P_{2|W|+1} = P_k \subseteq W \cup A_{3-i}$, we have $C_k^3 \subseteq (W \cup A_{3-i}) \bigotimes C_i \subseteq G$, a contradiction.

When k is even, $t = k/2 - 1$. It suffices to show only one class C_i has at most one edge. First notice that each C_i is P_3-free, since $(P_3 \cup P_{k-1}) \in \mathcal{M}(C_k^3)$ (see Figure 8(a)) and $P_{2|W|+1} = P_{k-1} \subseteq W \cup A_i$. Suppose some C_i has two isolated edges, say x_1y_1 and x_2y_2. Then each edge x_iy_i has at least one endpoint adjacent to some vertices in W, w.l.o.g. let them be x_1, x_2, since otherwise deleting x_iy_i and adding all edges between W and $\{x_i, y_i\}$ (at least two such edges) results in a C_k^3-free graph with more edges than G, a contradiction. If x_1, x_2 are adjacent to the same vertex $u \in W$, then y_1, x_1, u, x_2, y_2 form a P_5, and $P_{k-3} \in$

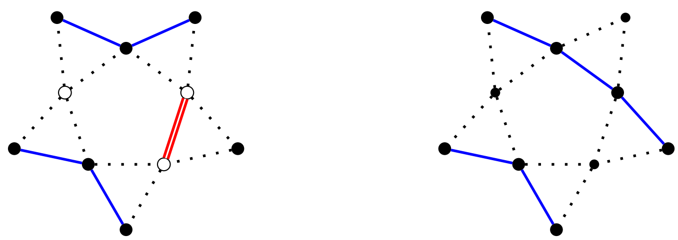


(a) (b)

Figure 8: (a): $(P_3 \cup P_{k-1}) \in \mathcal{M}(C_k^3)$ when $k = 4$; (b): $(P_5 \cup P_{k-3}) \in \mathcal{M}(C_k^3)$ when $k = 6$.

$(W - u) \cup (A_i - \{x_1, x_2, y_1, y_2\})$. Thus $(P_5 \cup P_{k-3}) \subseteq (W \cup C_i)$, a contradiction since $(P_5 \cup P_{k-3}) \in \mathcal{M}(C_k^3)$ (see Figure 8(b)). If x_1, x_2 are adjacent to different vertices in W, then a copy of P_{k+1} can be obtained in $W \cup C_i$ by prolonging a P_{k-1} using edges x_1y_1 and x_2y_2. Then we get a contradiction since $P_{k+1} \in \mathcal{M}(C_k^3)$. Thus each C_i has at most one edge. Now suppose both C_i, $i = 1, 2$, contain an edge u_iv_i with u_i adjacent to some vertices in W. Then similarly we can get a copy of P_k in $G[C_1 \cup W]$, starting at a vertex in A_1 and ending with edge u_1v_1. Let the vertices on this path be $x_1, x_2, \ldots, x_{k-2}, u_1, v_1$ with $x_1 \in A_1$. We can expand the path $x_1, \ldots, x_{k-2}, u_1$ to a copy of P_{k-1}^3. Since $x_1 \in A_1$, x_1 is adjacent to all vertices in C_2. In particular, x_1 is adjacent to both u_2 and v_2. Thus u_2, v_2 together with that copy of P^3_{k-1} form a P^3_k. Note that if there are at least three edges between $\{u_1, v_1\}$ and $\{u_2, v_2\}$, then it would complete P^3_k to a copy of C^3_k. Thus there are at most two edges, then delete u_2v_2 and add the missing edges between $\{u_1, v_1\}$ and $\{u_2, v_2\}$. The resulting graph is still C^3_k-free but with more edges than G, a contradiction. $\square$

Proof of Claim 22. We distinguish two cases depending on the parity of k.
Case 1: k is odd. First we show that H_i, $i = 1, 2$, is P_3-free. Suppose to the contrary that $P_3 \subseteq H_1$, then H_2 has to be an isolated vertex. Since otherwise $\frac{k-1}{2}P_3 \cup I_k \subseteq H_1$ and $P_2 \cup I_k \subseteq H_2$. This yields a contradiction since $C^3_k \subseteq (\frac{k-1}{2}P_3 \cup I_k)\bigotimes(P_2 \cup I_k)$ (see Figure 9(a)). Furthermore, since $(\frac{k-3}{2}P_3 \cup P_4) \in \mathcal{M}(C^3_k)$ (see Figure 9(b)), W is empty and H_1 is P_4-free, otherwise $(\frac{k-3}{2}P_3 \cup P_4) \subseteq W \cup C_1$, a contradiction. If $k = 3$, then $t = 1$,



(a) (b)

Figure 9: (a): $C^3_k \subseteq (\frac{k-1}{2}P_3 \cup I_k)\bigotimes(P_2 \cup I_k)$ when $k = 5$; (b): $(\frac{k-3}{2}P_3 \cup P_4) \in \mathcal{M}(C^3_k)$ when $k = 5$.

$K_3 \in \mathcal{M}(C^3_3)$, and H_1 is $\{K_3, P_4\}$-free, which implies H_1 is a star of constant order, say r. Then

$$e(G) \sim \frac{n^2}{4} + \frac{r-1}{r}\frac{n}{2} < \frac{n^2}{4} + \frac{nt}{p} = \frac{n^2}{4} + \frac{n}{2} \sim e(H(n, 2, 2)),$$

a contradiction. If $k \geq 5$, then $t \geq 2$. By Erdos-Gallai, since H_1 is P_4-free, the size of G is maximized when $H_1 = K_3$, hence

$$e(G) \sim \frac{n^2}{4} + \frac{n}{2} < \frac{n^2}{4} + n \leq \frac{n^2}{4} + \frac{nt}{p} \sim e(H(n, 2, t+1)),$$

a contradiction. Thus we may assume H_i, $i = 1, 2$, is P_3-free.

If $k = 3$ and one of H_i is not an isolated vertex, then W is empty, since otherwise $K_3 \subseteq W \cup H_i$. This implies for $i = 1, 2$, either $H_i = P_1$ and $|W| = 1$, or $H_i = P_2$ and $|W| = 0$, namely $G \simeq H(n, 2, 2)$ or $G \simeq H^*(n)$. Some calculation shows that when $4|n$, $H^*(n)$ has one more edge, otherwise they are of the same size.

If $k \geq 5$, then $t \geq 2$. If $H_1 = H_2 = P_2$, then since $C^3_k \subseteq (P_3 \cup \frac{k-3}{2}P_2 \cup I_k)\bigotimes(\frac{k-1}{2}P_2)$ (see Figure 10(a)), W is empty, otherwise $P_3 \cup \frac{k-3}{2}P_2 \cup I_k \subseteq W \cup C_1$. Thus $G \subseteq H^*(n)$, however this implies $e(G) \leq e(H^*(n)) \sim n^2/4 + n/2 < H(n, 2, t+1)$, a contradiction. Thus H_1 and H_2 can not both be P_2. Suppose $H_1 = P_2$ and $H_2 = P_1$. Define W to be the set of exceptional vertices that have neighbors in both A_1 and A_2. Then $|W| < t = \frac{k-1}{2}$, since otherwise $P_{k+1} \subseteq W \cup A_1$ and $P_{k+1} \in \mathcal{M}(C_k^3)$. Then

$$e(G) = \frac{n^2}{4} + \frac{t-1}{2}n + \frac{n}{4} + O(1) < \frac{n^2}{4} + \frac{t}{2}n + O(1) = e(H(n, 2, t+1)),$$

a contradiction. Thus $H_1 = H_2 = P_1$.

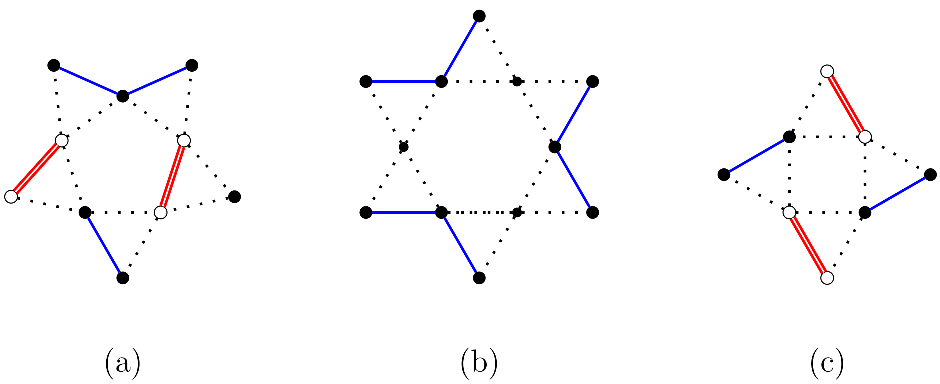


Figure 10: (a): $C_k^3 \subseteq (P_3 \cup \frac{k-3}{2}P_2 \cup I_k) \bigotimes (\frac{k-1}{2}P_2)$ when $k = 5$; (b): $\frac{k}{2}P_3 \in \mathcal{M}(C_k^3)$ when $k = 6$; (c): $C_k^3 \subseteq \frac{k}{2}P_2 \bigotimes \frac{k}{2}P_2$ when $k = 4$.

When k is even, since $\frac{k}{2}P_3 \in \mathcal{M}(C_k^3)$ (see Figure 10(b)), H_i is P_3-free, $i = 1, 2$. Also $C_k^3 \subseteq \frac{k}{2}P_2 \bigotimes \frac{k}{2}P_2$ (see Figure 10(c)), hence at most one H_i has an edge. W.l.o.g. suppose $H_1 = P_2$ and $H_2 = P_1$. Define W as before and similarly $|W| < t = \frac{k}{2} - 1$, which implies $e(G) < e(H(n, 2, t+1))$, a contradiction. Thus $H_1 = H_2 = P_1$. $\square$

For blow-ups of paths, notice that no matter what parity k is, $P_{k+1}^{p+1} \subseteq \lceil \frac{k}{2} \rceil P_2 \bigotimes \lceil \frac{k}{2} \rceil P_2$ and $P_{k+1}^{p+1} \subseteq \lceil \frac{k}{2} \rceil P_3 \bigotimes I_k$. With these two observations, the same argument works.

Proof of Theorem 5 (ii) The proof is quite similar to (i), except this time we make use of a forest in the decomposition family obtained by splitting a vertex of degree 2 in A. We include here only a sketch of the proof: proceed as in the proof of Theorem 5 (i), define W, A_i, B_i, C_i in the same way. Note that still $|W| = a - 1$. Since otherwise either the extremal graph G contains a forbidden graph (because some graph in the decomposition family shows up in $G[W \cup C_i]$ for some i) or it has fewer edges than $H'(n, p, a)$. It suffices to show that:

(a) every $G[C_i]$ can have at most one edge, and

(b) at most one $G[C_i]$ can have one such edge.

Let $F = T_1 \cup T_2 \in \mathcal{M}(T^{p+1})$ be the forest obtained by splitting some $z \in A$ with $d(z) = 2$. Let z_1 and z_2 be the two leaves corresponding to z after splitting it and define for $i = 1, 2$, $T_i' = T_i - z_i$.

We may assume $a = 2$, namely W is non-empty. Since otherwise this tree is a P_3, then for (a) two edges in some $G[C_i]$ form a linear forest of size two which is in $\mathcal{M}(P_3^{p+1})$, a contradiction; for (b), if $G[C_i]$ and $G[C_j]$, $i \neq j$, both contains an edge, then we are also done since $P_3^{p+1} \subseteq (P_2 \cup I_v) \bigotimes (P_2 \cup I_v) \bigotimes T_{n'',p-1}$ for sufficiently large v and n''.

For (a), suppose some $G[C_i]$ contains two edges e_1, e_2. Similar as Claim 19 (ii), e_1, e_2 each has at least one endpoint adjacent to W. If e_1, e_2 are disjoint, then notice that one can embed $T_1' \cup T_2'$ in $G[C_i \cup W]$ and get a copy of F by extending T_1' and T_2' using e_1 and e_2 respectively. This yields a contradiction since $G[C_i \cup W]$ is $\mathcal{M}T^{p+1}$-free. If e_1, e_2 share an endpoint, namely there is a $P_3 = \{w, x, y\}$ in $G[C_i]$. It is not hard to see that there is an embedding for T in $G[C_i \cup W]$, in which $A - \{z\}$ is embedded in W and z is embedded to x.

For (b), suppose for $i \neq j$, both $G[C_i]$ and $G[C_j]$ contain an edge e_i and e_j respectively. Then, using e_i and e_j, one can partition $W = W_1 \cup W_2$, s.t. $T_1 \subseteq G[C_i \cup W_1]$ and $T_2 \subseteq G[C_j \cup W_2]$, which yields a contradiction since $T^{p+1} \subseteq (T_1 \cup I_v) \bigotimes (T_2 \cup I_v) \bigotimes T_{n'',p-1}$. $\square$