The typical structure of maximal triangle-free graphs

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January 12, 2015

Abstract

Recently, settling a question of Erdős, Balogh and Petříčková showed that there are at most $2^{n^2/8+o(n^2)}$ n-vertex maximal triangle-free graphs, matching the previously known lower bound. Here we characterize the typical structure of maximal triangle-free graphs. We show that almost every maximal triangle-free graph G admits a vertex partition $X \cup Y$ such that $G[X]$ is a perfect matching and Y is an independent set.

Our proof uses the Ruzsa-Szemerédi removal lemma, the Erdős-Simonovits stability theorem, and recent results of Balogh-Morris-Samotij and Saxton-Thomason on characterization of the structure of independent sets in hypergraphs. The proof also relies on a new bound on the number of maximal independent sets in triangle-free graphs with many vertex-disjoint P_3's, which is of independent interest.

1 Introduction

Given a family of combinatorial objects with certain properties, a fundamental problem in extremal combinatorics is to describe the *typical* structure of these objects. This was initiated in a seminal work of Erdős, Kleitman, and Rothschild [13] in 1976. They proved that almost all triangle-free graphs on n vertices are bipartite, that is, the proportion of n-vertex triangle-free graphs that are not bipartite goes to zero as $n \to \infty$. Since then, various extensions of this theorem have been established. The typical structure of H-free graphs has been studied when H is a large clique [3, 19], H is a fixed color-critical subgraph [23], H is a finite family of subgraphs [2], and H is an induced subgraph [4]. For sparse H-free

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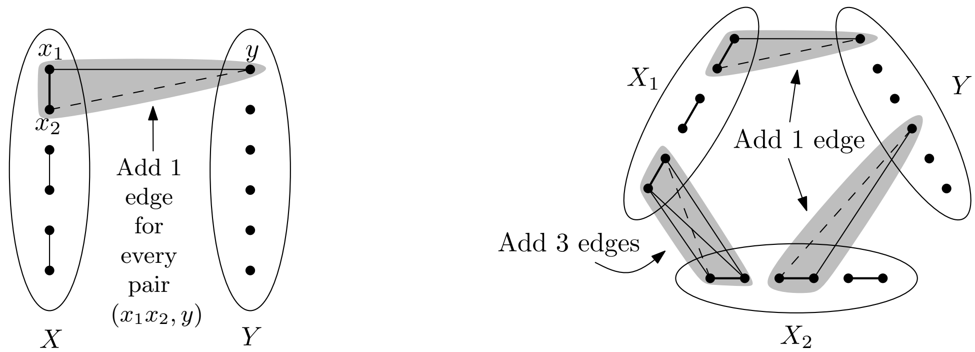


(a) $r = 2$ (b) $r = 3$

Figure 1: Lower bound contruction for maximal K_{r+1}-free graphs.

graphs, analogous problems were examined in [9, 21]. In the context of other combinatorial objects, the typical structure of hypergraphs with a fixed forbidden subgraph is investigated for example in [10, 22]; the typical structure of intersecting families of discrete structures is studied in [6]; see also [1] for a description of the typical sum-free set in finite abelian groups.

In contrast to the family of all n-vertex triangle-free graphs, which has been well-studied, very little was known about the subfamily consisting of all those that are maximal (under graph inclusion) triangle-free. Note that the size of the family of triangle-free graphs on $[n]$ is at least $2^{n^2/4}$ (all subgraphs of a complete balanced bipartite graph), and at most $2^{n^2/4+o(n^2)}$ by the result of Erdős, Kleitman, and Rothschild from 1976. Until recently, it was not even known if the subfamily of maximal triangle-free graphs is significantly smaller. As a first step, Erdős suggested the following problem (as stated in [26]): determine or estimate the number of maximal triangle-free graphs on n vertices. The following folklore construction shows that there are at least $2^{n^2/8}$ maximal triangle-free graphs on the vertex set $[n] := \{1, \dots, n\}$.

Lower bound construction. Assume that n is a multiple of 4. Start with a graph on a vertex set $X \cup Y$ with $|X| = |Y| = n/2$ such that X induces a perfect matching and Y is an independent set (see Figure 1a). For each pair of a matching edge x_1x_2 in X and a vertex $y \in Y$, add exactly one of the edges x_1y or x_2y. Since there are $n/4$ matching edges in X and $n/2$ vertices in Y, we obtain $2^{n^2/8}$ triangle-free graphs. These graphs may not be maximal triangle-free, but since no further edges can be added between X and Y, all of these $2^{n^2/8}$ graphs extend to distinct maximal ones.

Balogh and Petříčková [11] recently proved a matching upper bound, that the number of maximal triangle-free graphs on vertex set $[n]$ is at most $2^{n^2/8+o(n^2)}$. Now that the counting problem is resolved, one would naturally ask how do most of the maximal triangle-free graphs look, i.e. what is their typical structure. Our main result provides an answer to this question.

Theorem 1.1. *For almost every maximal triangle-free graph G on $[n]$, there is a vertex partition $X \cup Y$ such that $G[X]$ is a perfect matching and Y is an independent set.*

It is worth mentioning that once a maximal triangle-free graph has the above partition

$X \cup Y$, then there has to be exactly one edge between every matching edge of X and every vertex of Y. Thus Theorem 1.1 implies that almost all maximal triangle-free graphs have the same structure as the graphs in the lower bound construction above. Furthermore, our proof yields that the number of maximal triangle-free graphs without the desired structure is exponentially smaller than the number of maximal triangle-free graphs: Let $\mathcal{M}_3(n)$ denote the set of all maximal triangle-free graphs on $[n]$, and $\mathcal{G}(n)$ denote the family of graphs from $\mathcal{M}_3(n)$ that admit a vertex partition such that one part induces a perfect matching and the other is an independent set. Then there exists an absolute constant $c > 0$ such that for n sufficiently large, $|\mathcal{M}_3(n) - \mathcal{G}(n)| \leq 2^{-cn}|\mathcal{M}_3(n)|$.

It would be interesting to have similar results for $\mathcal{M}_r(n)$, the number of maximal K_r-free graphs on $[n]$. Alon pointed out that if the number of maximal K_r-free graphs is $2^{c_r n^2+o(n^2)}$, then c_r is monotone increasing in r, though not necessarily strictly monotone. For the lower bound, a discussion with Alon and Łuczak led to the following construction that gives $2^{(1-1/r+o(1))n^2/4}$ maximal K_{r+1}-free graphs: Assume that n is a multiple of $2r$. Partition the vertex set $[n]$ into r equal classes $X_1, \dots, X_{r-1}, Y$, and place a perfect matching into each of $X_1, \dots, X_{r-1}$ (see Figure 1b). Between the classes we have the following connection rule: between the vertices of two matching edges from different classes X_i and X_j place exactly three edges, and between a vertex in Y and a matching edge in X_i put exactly one edge. For the upper bound, by Erdős, Frankl and Rödl [12], $\mathcal{M}_{r+1}(n) \leq 2^{(1-1/r+o(1))n^2/2}$. A slightly improved bound is given in [11]: For every r there is $\varepsilon(r) > 0$ such that $|\mathcal{M}_{r+1}(n)| \leq 2^{(1-1/r-\varepsilon(r))n^2/2}$ for n sufficiently large. We suspect that the lower bound is the "correct value", i.e. that $|\mathcal{M}_{r+1}(n)| = 2^{(1-1/r+o(1))n^2/4}$.

Related problem. There is a surprising connection between the family of maximal triangle-free graphs and the family of maximal sum-free sets in $[n]$. More recently, Balogh, Liu, Sharifzadeh and Treglown [7] proved that the number of maximal sum-free sets in $[n]$ is $2^{(1+o(1))n/4}$, settling a conjecture of Cameron and Erdős. Although neither of the results imply one another, the methods in both of the papers fall in the same general framework, in which a rough structure of the family is obtained first using appropriate container lemma and removal lemma. These are Theorems 2.1 and 2.2 in this paper, and a group removal lemma of Green [16] and a granular theorem of Green and Ruzsa [17] in the sum-free case. Both problems can then be translated into bounding the number of maximal independent sets in some auxiliary link graphs. In particular, one of the tools here (Lemma 2.4) is also utilized in [8] to give an asymptotic of the number of maximal sum-free sets in $[n]$.

Organization. We first introduce all the tools in Section 2, then we prove Lemma 3.1, the asymptotic version of Theorem 1.1, in Section 3. Using this asymptotic result we prove Theorem 1.1 in Section 4.

Notation. For a graph G, denote by $|G|$ the number of vertices in G. An n-vertex graph G is *t-close to bipartite* if G can be made bipartite by removing at most t edges. Denote by P_k the path on k vertices. Write $\mathrm{MIS}(G)$ for the number of maximal independent sets in G. The *Cartesian product* $G \square H$ of graphs G and H is a graph with vertex set $V(G) \times V(H)$ such that two vertices (u, u') and (v, v') are adjacent if and only if either $u = v$ and $u'v' \in E(H)$, or $u' = v'$ and $uv \in E(G)$. For a fixed graph G, let $N(v)$ be the set of neighbors of a vertex v in G, and let $d(v) := |N_G(v)|$ and $\Gamma(v) := N(v) \cup \{v\}$. For $v \in V(G)$ and $X \subseteq V(G)$, denote by $N_X(v)$ the set of all neighbors of v in X (i.e. $N_X(v) = N(v) \cap X$), and let $d_X(v) := |N_X(v)|$. Denote by $\Delta(X)$ the maximum degree of the induced subgraph $G[X]$. Given a vertex partition $V = X_1 \cup X_2$, edges with one endpoint in X_1 and the other endpoint in X_2 are $[X_1, X_2]$-*edges*. A vertex cut $V = X \cup Y$ is a *max-cut* if the number of $[X, Y]$-edges is not smaller than the size of any other cut. The *inner neighbors* of a vertex v are its neighbors in the same partite set as v (i.e. $N_{X_i}(v)$ if $v \in X_i$). The *inner degree* of a vertex is the number of its inner neighbors. We say that a family $\mathcal{F}$ of maximal triangle-free graphs is *negligible* if there exists an absolute constant $C > 0$ such that $|\mathcal{F}| < 2^{-Cn}|\mathcal{M}_3(n)|$.

2 Tools

Our first tool is a corollary of recent powerful counting theorems of Balogh-Morris-Samotij [5, Theorem 2.2.], and Saxton-Thomason [25].

Theorem 2.1. *For all $\delta > 0$ there is $c = c(\delta) > 0$ such that there is a family $\mathcal{F}$ of at most $2^{c \cdot \log n \cdot n^{3/2}}$ graphs on $[n]$, each containing at most δn^3 triangles, such that for every triangle-free graph G on $[n]$ there is an $F \in \mathcal{F}$ such that $G \subseteq F$, where n is sufficiently large.*

The graphs in $\mathcal{F}$ in the above theorem will be referred to as *containers*. A weaker version of Theorem 2.1, which can be concluded from the Szemerédi Regularity Lemma, could be used instead of Theorem 2.1 here. The only difference is that the upper bound on the size of $\mathcal{F}$ is $2^{o(n^2)}$.

We need two well-known results. The first is the Ruzsa-Szemerédi triangle-removal lemma [24] and the second is the Erdős-Simonovits stability theorem [14]:

Theorem 2.2. *For every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ and $n_0(\varepsilon) > 0$ such that any graph G on $n > n_0(\varepsilon)$ vertices with at most δn^3 triangles can be made triangle-free by removing at most εn^2 edges.*

Theorem 2.3. *For every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ and $n_0(\varepsilon) > 0$ such that every triangle-free graph G on $n > n_0(\varepsilon)$ vertices with at least $\frac{n^2}{4} - \delta n^2$ edges can be made bipartite by removing at most εn^2 edges.*

We also need the following lemma, which is an extension of results of Moon-Moser [20] and Hujter-Tuza [18].

Lemma 2.4. *Let G be an n-vertex triangle-free graph. If G contains at least k vertex-disjoint P_3's, then*

$$\mathrm{MIS}(G) \leq 2^{\frac{n}{2} - \frac{k}{25}}. \tag{1}$$

Proof. The proof is by induction on n. The base case of the induction is $n = 1$ with $k = 0$, for which $\mathrm{MIS}(G) = 1 \leq 2^{\frac{1}{2}-\frac{0}{25}}$.

For the inductive step, let G be a triangle-free graph on $n \geq 2$ vertices with k vertex-disjoint P_3's, and let v be any vertex in G. Observe that $\mathrm{MIS}(G - \Gamma(v))$ is the number of maximal independent sets containing v, and that $\mathrm{MIS}(G - \{v\})$ bounds from above the number of maximal independent sets not containing v. Therefore,

$$\mathrm{MIS}(G) \leq \mathrm{MIS}(G - \{v\}) + \mathrm{MIS}(G - \Gamma(v)).$$

If G has k vertex-disjoint P_3's, then $G - \Gamma(v)$ has at least $k - d(v)$ vertex-disjoint P_3's, and so, by the induction hypothesis,

$$\mathrm{MIS}(G) \leq 2^{\frac{n-1}{2}-\frac{k-1}{25}} + 2^{\frac{n-(d(v)+1)}{2}-\frac{k-d(v)}{25}} \leq 2^{\frac{n}{2}-\frac{k}{25}} \left(2^{-\frac{1}{2}+\frac{1}{25}} + 2^{-\frac{d(v)+1}{2}+\frac{d(v)}{25}}\right).$$

The function $f(x) = 2^{-\frac{1}{2}+\frac{1}{25}} + 2^{-\frac{x+1}{2}+\frac{x}{25}}$ is a decreasing function with $f(3) \approx 0.9987 < 1$. So, if there exists a vertex of degree at least 3 in G, then we have $\mathrm{MIS}(G) \leq 2^{\frac{n}{2}-\frac{k}{25}}$ as desired.

It remains to verify (1) for graphs with $\Delta(G) \leq 2$. Observe that we can assume that G is connected. Indeed, if $G_1, \ldots, G_l$ are maximal components of G, and each of G_i has n_i vertices and k_i vertex-disjoint P_3's, then

$$\mathrm{MIS}(G) = \prod_i \mathrm{MIS}(G_i) \leq \prod_i 2^{\frac{n_i}{2}-\frac{k_i}{25}} = 2^{\sum_i \frac{n_i}{2} - \sum_i \frac{k_i}{25}} = 2^{\frac{n}{2}-\frac{k}{25}}.$$

Every connected graph with $\Delta(G) \leq 2$ and $n \geq 2$ vertices is either a path or a cycle. Suppose first that G is a path P_n. We have $\mathrm{MIS}(P_2) = 2 \leq 2^{\frac{2}{2}-\frac{0}{25}}$, $\mathrm{MIS}(P_3) = 2 \leq 2^{\frac{3}{2}-\frac{1}{25}}$. By Füredi [15, Example 1.1], $\mathrm{MIS}(P_n) = \mathrm{MIS}(P_{n-2}) + \mathrm{MIS}(P_{n-3})$ for all $n \geq 4$. By the induction hypothesis thus

$$\mathrm{MIS}(P_n) \leq 2^{\frac{n-2}{2}-\frac{k-1}{25}} + 2^{\frac{n-3}{2}-\frac{k-1}{25}} \leq 2^{\frac{n}{2}-\frac{k}{25}} \left(2^{-1+\frac{1}{25}} + 2^{-\frac{3}{2}+\frac{1}{25}}\right) \leq 2^{\frac{n}{2}-\frac{k}{25}}.$$

Let now G be a cycle C_n. We have $\mathrm{MIS}(C_4) = 2 \leq 2^{4/2-1/25}$ and $\mathrm{MIS}(C_5) = 5 \leq 2^{5/2-1/25}$. By Füredi [15, Example 1.2], $\mathrm{MIS}(C_n) = \mathrm{MIS}(C_{n-2}) + \mathrm{MIS}(C_{n-3})$ for all $n \geq 6$. Therefore, by the induction hypothesis,

$$\mathrm{MIS}(C_n) \leq 2^{\frac{n-2}{2}-\frac{k-1}{25}} + 2^{\frac{n-3}{2}-\frac{k-1}{25}} \leq 2^{\frac{n}{2}-\frac{k}{25}}.$$

□

Remark 2.5. A disjoint union of C_5's and a matching shows that the constant c for which $\mathrm{MIS}(G) \leq 2^{\frac{n}{2}-\frac{k}{c}}$ in Lemma 2.4 cannot be smaller than 5.6.

3 Asymptotic result

In this section we prove an asymptotic version of Theorem 1.1:

Lemma 3.1. *Fix any $\gamma > 0$. Almost every maximal triangle-free graph G on the vertex set $[n]$ satisfies the following: for any max-cut $V(G) = X \cup Y$, there exist $X' \subseteq X$ and $Y' \subseteq Y$ such that*

(i) $|X'| \leq \gamma n$ and $G[X - X']$ is an induced perfect matching, and

(ii) $|Y'| \leq \gamma n$ and $Y - Y'$ is an independent set.

The outline of the proof is as follows. We observe that every maximal triangle-free graph G on $[n]$ can be built in the following three steps.

(S1) Choose a max-cut $X \cup Y$ for G.

(S2) Choose triangle-free graphs S and T on the vertex sets X and Y, respectively.

(S3) Extend $S \cup T$ to a maximal triangle-free graph by adding edges between X and Y.

We give an upper bound on the number of choices for each step. First, there are at most 2^n ways to fix a max-cut $X \cup Y$ in (S1). For (S2), we show (Lemma 3.5) that almost all maximal triangle-free graphs on $[n]$ are $o(n^2)$-close to bipartite, which implies that the number of choices for most of these graphs in (S2) is at most $2^{o(n^2)}$. For fixed X, Y, S, T, we bound, using Claim 3.4, the number of choices in (S3) by the number of maximal independent sets in some auxiliary link graph L. This enables us to use Lemma 2.4 to force the desired structure on S and T.

Definition 3.2 (Link graph). Given edge-disjoint graphs A and S on $[n]$, define the *link graph* $L := L_S[A]$ of S on A as follows:

$$V(L) := E(A) \quad \text{and} \quad E(L) := \{a_1a_2 : \exists s \in E(S) \text{ such that } \{a_1, a_2, s\} \text{ forms a triangle}\}.$$

Claim 3.3. *If A and S are triangle-free, then $L_S[A]$ is triangle-free.*

Proof. Indeed, otherwise there exist $a_1, a_2, a_3 \in E(A)$ and $s_1, s_2, s_3 \in E(S)$ such that the 3-sets $\{a_1, a_2, s_1\}$, $\{a_2, a_3, s_2\}$, and $\{a_1, a_3, s_3\}$ span triangles. Since A is triangle-free, the edges a_1, a_2, a_3 share a common endpoint, and $\{s_1, s_2, s_3\}$ spans a triangle. This is a contradiction since S is triangle-free. □

Claim 3.4. *Let S and A be two edge-disjoint triangle-free graphs on $[n]$ such that there is no triangle $\{a, s_1, s_2\}$ in $S \cup A$ with $a \in E(A)$ and $s_1, s_2 \in E(S)$. Then the number of maximal triangle-free subgraphs of $S \cup A$ containing S is at most* $\mathrm{MIS}(L_S[A])$.

Proof. Let G be a maximal triangle-free subgraph of $S \cup A$ that contains S. We show that $E(G) \cap E(A)$ spans a maximal independent set in $L := L_S[A]$. Clearly, $E(G) \cap E(A)$ spans an independent set in L because otherwise there would be a triangle in G. Suppose that $E(G) \cap E(A)$ is not a maximal independent set in L. Then there is $a_1 \in E(A) - E(G)$ such that, for any two edges $a_2 \in E(A) \cap E(G)$ and $s \in E(S)$, $\{a_1, a_2, s\}$ does not form a triangle. By our assumption, there is no triangle $\{a_1, a_2, a_3\}$ with $a_2, a_3 \in E(A)$ and no triangle $\{a_1, s_1, s_2\}$ with $s_1, s_2 \in E(S)$. Therefore, $G \cup \{a_1\}$ is triangle-free, contradicting the maximality of G. □

We fix the following parameters that will be used throughout the rest of the paper. Let $\gamma, \beta, \varepsilon, \varepsilon' > 0$ be sufficiently small constants satisfying the following hierachy:

$$\varepsilon' \ll \delta_{2.3}(\varepsilon) \ll \varepsilon \ll \beta \ll \delta_{2.3}(\gamma^3) \ll \gamma \ll 1, \tag{2}$$

where $\delta_{2.3}(x) > 0$ is the constant returned from Theorem 2.3 with input x. The notation $x \ll y$ above means that x is a sufficiently small function of y to satisfy some inequalities in the proof. In the following proof, $\delta_{2.2}(x)$ is the constant returned from Theorem 2.2 with input x, and in the rest of the paper, we shall always assume that n is sufficiently large, even when this is not explicitly stated.

Lemma 3.5. *Almost all maximal triangle-free graphs on $[n]$ are $2\varepsilon n^2$-close to bipartite.*

Proof. Let $\mathcal{F}$ be the family of graphs obtained from Theorem 2.1 using $\delta_{2.2}(\varepsilon')$. Then every triangle-free graph on $[n]$ is a subgraph of some container $F \in \mathcal{F}$.

We first show that the family of maximal triangle-free graphs in small containers is negligible. Consider a container $F \in \mathcal{F}$ with $e(F) \leq n^2/4 - 6\varepsilon' n^2$. Since F contains at most $\delta_{2.2}(\varepsilon')n^3$ triangles, by Theorem 2.2, we can find A and B, subgraphs of F, such that $F = A \cup B$, where A is triangle-free, and $e(B) \leq \varepsilon' n^2$. For each $F \in \mathcal{F}$, fix such a pair (A, B). Then every maximal triangle-free graph in F can be built in two steps:

- (i) Choose a triangle-free $S \subseteq B$;
- (ii) Extend S in A to a maximal triangle-free graph.

The number of choices in (i) is at most $2^{e(B)} \leq 2^{\varepsilon' n^2}$. Let $L := L_S[A]$ be the link graph of S on A. By Claim 3.3, L is triangle-free. Claim 3.4 implies that the number of maximal triangle-free graphs in $S \cup A$ containing S (i.e. the number of extensions in (ii)) is at most $\text{MIS}(L)$. Thus, by Lemma 2.4,

$$\text{MIS}(L) \leq 2^{|A|/2} \leq 2^{n^2/8 - 3\varepsilon' n^2}.$$

Therefore, the number of maximal triangle-free graphs in small containers is at most

$$|\mathcal{F}| \cdot 2^{\varepsilon' n^2} \cdot 2^{n^2/8 - 3\varepsilon' n^2} \leq 2^{n^2/8 - \varepsilon' n^2}.$$

From now on, we may consider only maximal triangle-free graphs contained in containers of size at least $n^2/4 - 6\varepsilon' n^2$. Let F be any large container. Recall that by Theorem 2.2, $F = A \cup B$, where A is triangle-free with $e(A) \geq n^2/4 - 7\varepsilon' n^2$ and $e(B) \leq \varepsilon' n^2$. Since $\varepsilon' \ll \delta_{2.3}(\varepsilon)$, by Theorem 2.3, A can be made bipartite by removing at most εn^2 edges. Since $\varepsilon' \ll \varepsilon$, F can be made bipartite by removing at most $(\varepsilon' + \varepsilon)n^2 \leq 2\varepsilon n^2$ edges. Therefore, every maximal triangle-free graphs contained in F is $2\varepsilon n^2$-close to bipartite. □

Fix X, Y, S, T as in steps (S1) and (S2). Let A be the complete bipartite graph with parts X and Y. By Claim 3.4, the number of ways to extend $S \cup T$ in (S3) is at most $\text{MIS}(L_{S \cup T}[A])$. The number of ways to fix X and Y is at most 2^n, and by Lemma 3.5, the number of ways to fix S and T is at most $\binom{n^2}{2\varepsilon n^2}$. It follows that if $\text{MIS}(L_{S \cup T}[A])$ is smaller than $2^{n^2/8 - cn^2}$ for some $c \gg \varepsilon$, then the family of maximal triangle-free graphs with such (X, Y, S, T) is negligible.

Claim 3.6. $L_{S\cup T}[A] = S\square T$.

Proof. Note that $V(L_{S\cup T}[A]) = E(A) = \{(x,y) : x \in X, y \in Y\} = V(S\square T)$. Using the definition of the Cartesian product, (x,y) and (x',y') are adjacent in $S\square T$ if and only if $x = x'$ and $\{y,y'\} \in E(T)$, or $y = y'$ and $\{x,x'\} \in E(S)$, i.e. if and only if $\{x = x', y, y'\}$ or $\{x, x', y = y'\}$ form a triangle in $S \cup A$. But by the definition of $L_{S\cup T}[A]$, this is exactly when (x,y) and (x',y') are adjacent in $L_{S\cup T}[A]$. $\square$

Claim 3.6 allows us to rule out certain structures of S and T since, by Lemma 2.4, if $S\square T$ has many vertex disjoint P_3's then the number of maximal-triangle free graphs with $S = G[X]$ and $T = G[Y]$ is much smaller than $2^{n^2/8}$.

Claim 3.7. *For almost all maximal triangle-free n-vertex graphs G with a max-cut $X \cup Y$,*
(i) $|X|, |Y| \geq n/2 - \beta n$, *and*
(ii) $\Delta(X), \Delta(Y) \leq \beta n$.

Proof. Let G be a maximal triangle-free graph with a max-cut $X\cup Y$. By Lemma 3.5, almost all maximal triangle-free graphs are $2\varepsilon n^2$-close to bipartite, which implies that the number of choices for $G[X]$ and $G[Y]$ is at most $\binom{n^2}{2\varepsilon n^2}$. Denote by A the complete bipartite graph with partite sets X and Y.

For (i), suppose that $|X| \leq n/2 - \beta n$. Then $|X||Y| \leq n^2/4 - \beta^2 n^2$, and for any fixed S on X and T on Y, Lemma 2.4 implies $\mathrm{MIS}(L_{S\cup T}[A]) \leq 2^{n^2/8-\beta^2n^2/2}$. Since $\beta \gg \varepsilon$, it follows from the discussion before Claim 3.6 that the family of maximal triangle-free graphs with such max-cut $X \cup Y$ is negligible.

For (ii), suppose that G has a vertex $x \in X$ of inner degree at least βn. Since $X \cup Y$ is a max-cut, $|N_Y(x)| \geq |N_X(x)| \geq \beta n$. Since G is triangle-free, there is no edge in between $N_X(x)$ and $N_Y(x)$. Let $A' \subseteq A$ be a graph formed by deleting all edges between $N_X(x)$ and $N_Y(y)$ from A. Define a link graph $L' := L_{S\cup T}[A']$ of $S \cup T$ on A'. In this case, the number of choices for (S3) is at most $\mathrm{MIS}(L')$. Since L' is triangle-free (Claim 3.3) and $|L'| = e(A') \leq |X||Y| - |N_X(x)||N_Y(x)| \leq \frac{n^2}{4} - \beta^2 n^2$, it follows from Lemma 2.4 that

$$\mathrm{MIS}(L') \leq 2^{|L'|/2} \leq 2^{n^2/8-\beta^2n^2/2}.$$

$\square$

Proof of Lemma 3.1. First, we show that for almost every maximal triangle-free graph G on $[n]$ with max-cut $X \cup Y$ and with $G[X] = S$ and $G[Y] = T$, there are very few vertex-disjoint P_3's in $S \cup T$. Suppose that there exist βn vertex-disjoint P_3's in S or in T, say in S. Since $L_{S\cup T}[A] = S\square T$ by Claim 3.6, and for each of the βn vertex-disjoint P_3's in S we obtain $|T|$ vertex-disjoint P_3's in $S\square T$, the number of vertex-disjoint P_3's in $L_{S\cup T}[A]$ is at least $\beta n|T| = \beta n|Y|$. By Claim 3.7(i), $\beta n|Y| \geq \beta n(n/2 - \beta n) \geq \beta n^2/3$. Then by Lemma 2.4,

$$\mathrm{MIS}(L_{S\cup T}[A]) \leq 2^{|S\square T|/2-\beta n^2/75} \leq 2^{n^2/8-\beta n^2/75}.$$

Since $\beta \gg \varepsilon$, the family of maximal triangle-free graphs with such (X, Y, S, T) is negligible. Hence, for almost every maximal triangle-free graph G with some (X, Y, S, T), we can find

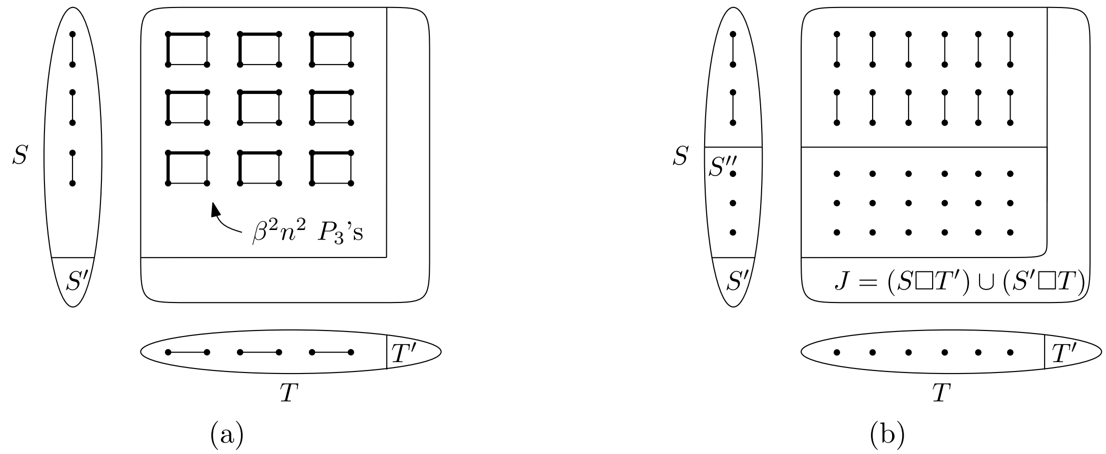


Figure 2: Forbidden structures in S and T.

some induced subgraphs $S' \subseteq S$ and $T' \subseteq T$ with $|S'| \leq 3\beta n$ and $|T'| \leq 3\beta n$ such that both $S - S'$ and $T - T'$ are P_3-free. This implies that each of $S - S'$ and $T - T'$ is a union of a matching and an independent set.

Next, we show that at most one of the graphs S and T can have a large matching. Suppose both S and T have a matching of size at least βn, then there are at least $\beta^2 n^2$ vertex-disjoint C_4's in $S\Box T$, each of which contains a copy of P_3 (see Figure 2a). It follows that the family of such graphs is negligible since $\mathrm{MIS}(L_{S\cup T}[A]) \leq 2^{n^2/8-\beta^2 n^2/25}$ and $\beta \gg \varepsilon$. Hence, we can assume that all but $2\beta n$ vertices in T form an independent set. Redefine T' so that $|T'| \leq 2\beta n$ and $V(T - T')$ is an independent set.

Lastly, we show that there are very few isolated vertices in the graph $S - S'$. Suppose that there are $\gamma n/2$ isolated vertices in $S - S'$, spanning a subgraph S'' of S. We count $\mathrm{MIS}(S\Box T)$ as follows. Let $J := (S\Box T') \cup (S'\Box T)$ and $L' := S\Box T - J$. Every maximal independent set in $S\Box T$ can be built by

(i) choosing an independent set in J, and
(ii) extending it to a maximal independent set in L'.

Since $|J| \leq |S'||T| + |T'||S| \leq 3\beta n \cdot n + 2\beta n \cdot n = 5\beta n^2$, there are at most $2^{|J|} = 2^{5\beta n^2}$ choices for (i). Note that L' consists of isolated vertices from $S''\Box(T - T')$ and an induced matching from $(S - S' - S'')\Box(T - T')$ (see Figure 2b). Thus the number of extensions in (ii) is at most $\mathrm{MIS}((S - S' - S'')\Box(T - T'))$. The graph $(S - S' - S'')\Box(T - T')$ is a perfect matching with

$$\frac{1}{2}|S - S' - S''||T - T'| \leq \frac{1}{2}|S - S''||T| \leq \frac{1}{2}\left(|S| - \frac{\gamma n}{2}\right)(n - |S|) \leq \frac{1}{2}\left(\frac{n}{2} - \frac{\gamma n}{4}\right)^2 \leq \frac{n^2}{8} - \frac{\gamma n^2}{16}$$

edges, and so choosing one vertex for each matching edge gives at most $2^{n^2/8-\gamma n^2/16}$ maximal independent sets. Since $\beta \ll \gamma$, it follows that $\mathrm{MIS}(S\Box T) \leq 2^{5\beta n^2} \cdot 2^{n^2/8-\gamma n^2/16} \leq 2^{n^2/8-\gamma n^2/17}$. Thus, such family of maximal triangle-free graphs is negligible, and we may assume that $|S''| \leq \gamma n/2$.

The statement of Lemma 3.1 follows by setting $X' := V(S' \cup S'')$ and $Y' := V(T')$. Indeed, $|X'| \leq 3\beta n + \gamma n/2 \leq \gamma n$, $|Y'| \leq 2\beta n \leq \gamma n$, $G[X - X'] = S - S' - S''$ is a perfect matching, and $Y - Y' = V(T) - V(T')$ is an independent set.

□

4 Proof of Theorem 1.1

For the proof of Theorem 1.1, we need to introduce several classes of graphs on the vertex set $V = [n]$. Recall the hierarchy of parameters fixed in Section 3:

$$\varepsilon' \ll \delta_{2.3}(\varepsilon) \ll \varepsilon \ll \beta \ll \delta_{2.3}(\gamma^3) \ll \gamma \ll 1, \tag{3}$$

Definition 4.1. Fix a vertex partition $V = X \cup Y$, a perfect matching M on the vertex set X (in case $|X|$ is odd, M is an almost perfect matching covering all but one vertex of X), and non-negative integers r, s and t.

1. Denote by $\mathcal{B}(X, Y, M, s, t)$ the class of maximal triangle-free graphs G with max-cut $X \cup Y$ satisfying the following three conditions:
   (i) The subgraph $G[X]$ has a maximum matching $M' \subseteq M$ covering all but at most γn vertices in X;
   (ii) The size of a largest family of vertex-disjoint P_3's in $S := G[X]$ is s;
   (iii) The size of a maximum matching in $T := G[Y]$ is t.

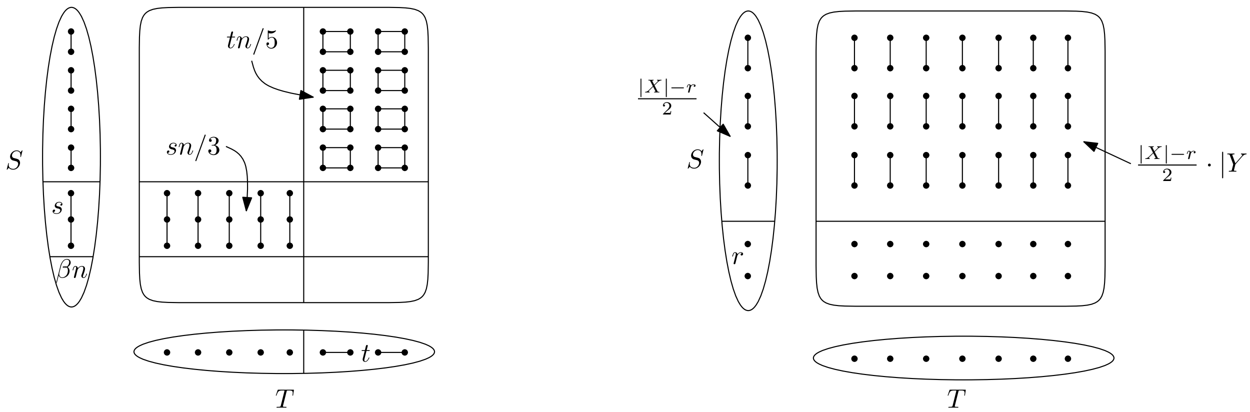
2. Denote by $\mathcal{B}(X, Y, M, r) \subseteq \mathcal{B}(X, Y, M, 0, 0)$ the subclass consisting of all graphs in $\mathcal{B}(X, Y, M, 0, 0)$ with exactly r isolated vertices in $G[X]$.

3. When $|X|$ is even, denote by $\mathcal{G}(X, Y, M)$ the class of all maximal triangle-free graphs G with max-cut $X \cup Y$, $G[X] = M$, and Y an independent set.

4. When $|X|$ is even, denote by $\mathcal{H}(X, Y, M)$ the class of maximal triangle-free graphs G that are constructed as follows:
   (P1) Add M to X;
   (P2) For every edge $x_1x_2 \in M$ and every vertex $y \in Y$, add either the edge x_1y or x_2y;
   (P3) Extend each of the $2^{|X||Y|/2}$ resulting graphs to a maximal triangle-free graph by adding edges in X and/or Y.

By Lemmas 3.1, 3.5 and Claim 3.7, throughout the rest of the proof, we may only consider maximal triangle-free graphs in $\bigcup_{X,Y,M,s,t} \mathcal{B}(X, Y, M, s, t)$ that are βn^2-close to bipartite, $|X|, |Y| \geq n/2 - \beta n$ and $\Delta(X), \Delta(Y) \leq \beta n$. We may further assume from the proof of Lemma 3.1 that $s, t \leq \beta n$.

Notice that graphs from $\mathcal{G}(X, Y, M) = \mathcal{B}(X, Y, M, 0)$ are precisely those with the desired structure. We will show that the number of graphs without the desired structure is exponentially smaller. The set of "bad" graphs consists of the following two types:
   (i) when $|X|$ is even, $\bigcup_{s,t} \mathcal{B}(X, Y, M, s, t) - \mathcal{B}(X, Y, M, 0)$;
   (ii) when $|X|$ is odd, $\bigcup_{s,t} \mathcal{B}(X, Y, M, s, t)$.



(a) The number of vertex-disjoint P_3's in $S\square T$ is at least $sn/3 + tn/5$ (Lemma 4.2).

(b) $\mathrm{MIS}(S\square T) \leq 2^{(|X|-r)|Y|/2}$ if $s = t = 0$ and X has r isolated vertices (Lemma 4.3).

Figure 3

Fix an arbitrary choice of (X, Y, M). For simplicity, let $\mathcal{B}(s,t) := \mathcal{B}(X,Y,M,s,t)$ and $\mathcal{B}(r) := \mathcal{B}(X,Y,M,r)$. Let A be the complete bipartite graph with parts X and Y.

Lemma 4.2. *If $s + t \geq 1$, then $|\mathcal{B}(s,t)| \leq 2^{|X||Y|/2 - n/200}$.*

Proof. Let s and t be two non-negative integers, at least one of which is nonzero. We first bound the number of ways to choose S and T, i.e. the number of ways to add inner edges. The number of ways to choose the vertex set of the s vertex-disjoint P_3's in S and the t matching edges in T is at most $\binom{n}{3s}\binom{n}{2t}$. Since $\Delta(X), \Delta(Y) \leq \beta n$, each of the $3s + 2t$ chosen vertices has inner degree at most βn. Therefore, the number of ways to choose their inner neighbors is at most

$$\binom{n}{\beta n}^{3s+2t} \leq \left(\left(\frac{en}{\beta n}\right)^{\beta n}\right)^{3s+2t} \leq 2^{\beta \log(e/\beta)\cdot(3s+2t)n}.$$

The number of ways to add the $[X,Y]$-edges is $\mathrm{MIS}(L_{S\cup T}(A))$. We claim that the link graph $L := L_{S\cup T}(A) = S\square T$ has at least $(s+t)n/5$ vertex-disjoint P_3's. Indeed, recall that $|S| = |T| \geq n/2 - \beta n$ and $s, t \leq \beta n$, thus in $S\square T$ (see Figure 3a), we have at least $s(|T| - 2t) \geq sn/3$ vertex-disjoint P_3's coming from s vertex-disjoint P_3's in S and at least $\frac{1}{2}(|S| - \beta n - 3s)\cdot t \geq tn/5$ vertex-disjoint P_3's coming from the Cartesian product of a matching in S and a matching in T. So by Lemma 2.4,

$$\mathrm{MIS}(L) \leq 2^{|X||Y|/2 - (s+t)n/125}.$$

Since $s + t \geq 1$ and β is sufficiently small,

$$|\mathcal{B}(s,t)| \leq \binom{n}{3s}\binom{n}{2t}\cdot 2^{\beta\log(e/\beta)\cdot(3s+2t)n}\cdot 2^{|X||Y|/2-(s+t)n/125} \leq 2^{|X||Y|/2-n/200}.$$

$\square$

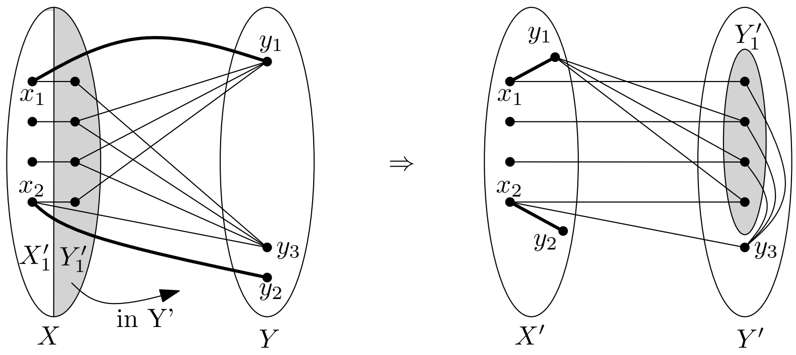


Figure 4: (X', Y', M') is uniquely determined after choosing $x_1y_1 \in M'$ (Lemma 4.4).

Lemma 4.3. *If $s = t = 0$ and $r \in \mathbb{Z}^+$, then $|\mathcal{B}(r)| \le 2^{|X||Y|/2-n/6}$.*

Proof. By the definition of $\mathcal{B}(r)$, X consists of r isolated vertices and a matching of size $(|X|-r)/2$, and Y is an independent set. Hence the graph $L_{S\cup T}(A) = S\square T$ consists of a matching of size $(|X|-r)|Y|/2$ and isolated vertices (see Figure 3b). There are at most $\binom{n}{r}$ ways to pick the isolated vertices in X and at most $\mathrm{MIS}(L_{S\cup T}(A))$ ways to choose the $[X, Y]$-edges. Recall that $|Y| \ge n/2 - \beta n$. Thus we have

$$|\mathcal{B}(r)| \le \binom{n}{r} \cdot 2^{(|X|-r)|Y|/2} \le 2^{|X||Y|/2+r\log n - rn/5} \le 2^{|X||Y|/2-rn/6} \le 2^{|X||Y|/2-n/6}.$$

□

Case 1: $|X|$ is even. For simplicity, denote $\mathcal{G} := \mathcal{G}(X, Y, M)$ and $\mathcal{H} := \mathcal{H}(X, Y, M)$.

Lemma 4.4. *An n-vertex graph G is in at most n^2 different classes $\mathcal{G}(X, Y, M)$.*

Proof. Let $G \in \mathcal{G}(X, Y, M)$. Recall that $G[X] = M$ and Y is an independent set. Thus G can be in a different class $\mathcal{G}(X', Y', M')$ if and only if $X' \ne X$, $Y' \ne Y$ and $M' \ne M$. Since $M' \ne M$ and Y is an independent set, there exists an edge x_1y_1 in M' with $x_1 \in X$ and $y_1 \in Y$. There are at most n^2 ways to choose such an edge. Since G is a maximal triangle-free graph, every vertex in Y is adjacent to exactly one vertex from each edge in M. Let x_1' be the neighbor of x_1 in X, and set $Y_1' := N_X(y_1) \cup \{x_1'\} - \{x_1\}$ and $X_1' = X - Y_1'$. Note that $x_1y_1 \in M'$ and $G[X'] = M'$ imply $Y_1' \subseteq Y'$. Since Y' is an independent set, it follows that $X_1' \subseteq X'$.

We claim that for any vertex $x_2 \in X_1'$, there is at most one vertex in Y that can serve as its neighbor in M' (see Figure 4). Suppose to the contrary that there are two such vertices y_2 and y_3 in Y. Then neither of y_2 and y_3 has neighbors in $X_1' - \{x_2\}$, and so both y_2 and y_3 are adjacent to all but one (the neighbor of x_2) vertex of $Y_1' \subseteq Y'$. If now $x_2y_2 \in M'$, then $y_3 \in Y'$. But y_3 is adjacent to some vertices of Y', which contradicts the independence of Y'. In conclusion, after we pick one of the edges of M' with exactly one end in X and one end in Y, since the graph G is labeled, the rest of X', Y' and M' is uniquely determined. □

By Lemma 4.4, it is sufficient to show that for any choice of (X, Y, M) with $|X|$ even,

$$\frac{|\bigcup_{s,t} \mathcal{B}(X, Y, M, s, t) - \mathcal{B}(X, Y, M, 0)|}{|\mathcal{G}(X, Y, M)|} \le 2^{-n/300}. \tag{4}$$

Lemma 4.5. *We have* $|\mathcal{G}| \ge (1 + o(1))2^{|X||Y|/2}$.

Proof. Recall that $|X|, |Y| \ge n/2 - \beta n$, and therefore $|\mathcal{H}| = 2^{|X||Y|/2} \gg 2^{n^2/8-\beta n^2}$. Running the same proof as Lemma 3.5 (start the proof by invoking Theorem 2.1 with $\delta_{2.2}(\beta)$, replace ε' by β and ε by γ^3) implies that almost all graphs in $\mathcal{H}$ are $2\gamma^3 n^2$-close to bipartite. Let $\mathcal{H}' \subseteq \mathcal{H}$ be the subfamily consisting of all those that are $2\gamma^3 n^2$-close to bipartite. Then it is sufficient to show $|\mathcal{H}' - \mathcal{G}| = o(2^{|X||Y|/2})$. There are two types of graphs in $\mathcal{H}' - \mathcal{G}$:

(i) $\mathcal{H}_1$: those that $X \cup Y$ is not one of its max-cut;

(ii) $\mathcal{H}_2$: those with $X \cup Y$ as a max-cut, but not maximal after (P2), i.e. there are inner edges added in X and/or Y in (P3).

We first bound the number of graphs in $\mathcal{H}_1$. Let $G \in \mathcal{H}_1$ with a max-cut $X' \cup Y'$ minimizing $|X \triangle X'|$. We may assume that $|X'|, |Y'| \ge n/2 - \gamma n$ and $\Delta(X'), \Delta(Y') \le \gamma n$. Indeed, since graphs in $\mathcal{H}_1$ are $2\gamma^3 n^2$-close to bipartite, if $|X'| \le n/2 - \gamma n$ or $\Delta(X') \ge \gamma n$, then the same proof as the proof of Claim 3.7 yields that the number of such graphs in $\mathcal{H}_1$ is at most

$$\binom{n^2}{2\gamma^3 n^2} \cdot 2^{n^2/8-\gamma^2 n^2/2} \le 2^{2\gamma^3 n^2 \log(e/2\gamma^3)} 2^{n^2/8-\gamma^2 n^2/2} \ll 2^{n^2/8-\gamma n^2},$$

which is exponentially smaller than $|\mathcal{H}'| = (1 + o(1))2^{|X||Y|/2} \gg 2^{n^2/8-\beta n^2}$.

Let $X_1 := X \cap X'$, $X_2 := X - X_1$, $Y_1 := Y \cap X'$, and $Y_2 := Y - Y_1$ (see Figure 5a). Since $X \cup Y$ is not a max-cut of G, the set $X \triangle X' = Y \triangle Y' = X_2 \cup Y_1$ is non-empty. By symmetry, we can assume that $Y_1 \neq \emptyset$. Recall that from (P2), for every $y \in Y_1 \subseteq X'$, we have $d_X(y) = |X|/2 \ge n/4 - \beta n/2$. It follows that $|X_2| \ge n/4 - 2\gamma n$, since otherwise $d_{X'}(y) \ge d_{X_1}(y) = d_X(y) - |X_2| \ge 3\gamma n/2$, contradicting $\Delta(X') \le \gamma n$. Similarly, we have $|X_1| \ge n/4 - 2\gamma n$. Recall also that $|X| = n - |Y| \le n/2 + \beta n$. Thus for $i = 1, 2$,

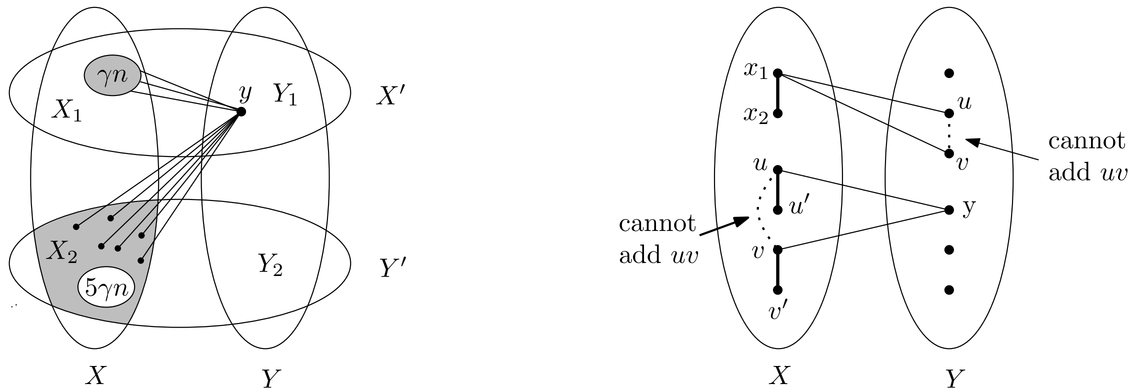
$$|X_i| = |X| - |X_{3-i}| \le \frac{n}{2} + \beta n - \left(\frac{n}{4} - 2\gamma n\right) \le \frac{n}{4} + 3\gamma n.$$

Therefore, for $i = 1, 2$, every vertex $y \in Y_i$ is adjacent to at most γn vertices in X_i and all but at most

$$|X_{3-i}| - (d_X(y) - d_{X_i}(y)) \le \frac{n}{4} + 3\gamma n - \left(\frac{n}{4} - \frac{\beta n}{2}\right) + \gamma n \le 5\gamma n$$

vertices in X_{3-i}, as shown in Figure 5a. Hence, for fixed X_1 and X_2, the number of ways to choose $N(y)$ for any $y \in Y$ is at most $\binom{|X_1|}{5\gamma n}\binom{|X_2|}{5\gamma n}$. Since the number of graphs in $\mathcal{H}_1$ is precisely the number of ways to add the $[X, Y]$-edges in (P2), we have

$$|\mathcal{H}_1| \le 2^{|X|} \cdot 2^{|Y|} \cdot \left(\binom{|X_1|}{5\gamma n}\binom{|X_2|}{5\gamma n}\right)^{|Y|} \le 2^{\gamma^{1/2} n^2},$$



(a) The structure of graphs in $\mathcal{H}_1$ with max-cut $X' \cup Y'$.

(b) Two examples of uv that cannot be added when forming a graph from $\mathcal{H}_2$.

Figure 5

where the first two terms count the number of ways to partition $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$, and the last term bounds the number of ways to choose the $[X, Y]$-edges.

We now bound the number of graphs in $\mathcal{H}_2$. For any graph $G \in \mathcal{H}_2$, some inner edges were added in (P3). Suppose that $[X, Y]$-edges added in (P2) were chosen randomly (each of $x_1 y$ and $x_2 y$ with probability $1/2$). Clearly, uv can be added in (P3) if and only if u and v have no common neighbor. Consider the case when $u, v \in X$ and let uu', vv' be the corresponding edges in M (see Figure 5b). Every $y \in Y$ is adjacent to exactly one of u, u' and exactly one of v, v'. Thus the probability that y is a common neighbor of u and v is $1/4$, which implies that uv can be added with probability $(3/4)^{|Y|}$. Let now $u, v \in Y$. Then u and v have no common neighbor if and only if for every $x_1 x_2 \in M$, u and v chose different neighbors among x_1 and x_2. So in this case we can add u, v with probability $(1/2)^{|X|/2}$. Summing over all possible outcomes of (P2) and all possible choices for uv implies

$$|\mathcal{H}_2| \leq 2^{|X||Y|/2} \cdot \binom{n}{2} \cdot \left(\left(\frac{1}{2}\right)^{|X|/2} + \left(\frac{3}{4}\right)^{|Y|} \right) \ll 2^{|X||Y|/2 - n/5}.$$

Hence, we have

$$|\mathcal{H}' - \mathcal{G}| = |\mathcal{H}_1| + |\mathcal{H}_2| \leq 2^{\gamma^{1/2} n^2} + 2^{|X||Y|/2 - n/5} = o(2^{|X||Y|/2}).$$

□

Since $s, t, r \leq n$, Lemmas 4.2, 4.3 and 4.5 imply (4):

$$\frac{\left| \bigcup_{s,t} \mathcal{B}(s,t) - \mathcal{B}(0) \right|}{|\mathcal{G}|} = \frac{\sum_{s,t:\ s+t \geq 1} |\mathcal{B}(s,t)| + \sum_{r \geq 1} |\mathcal{B}(r)|}{|\mathcal{G}|} \leq \frac{(n^2 + n) \cdot 2^{|X||Y|/2 - n/200}}{(1 + o(1)) 2^{|X||Y|/2}} \leq 2^{-n/300}.$$

Case 2: $|X|$ is odd.

Fix an arbitrary choice of X, Y, M with $|X|$ odd and let $x \in X$ be the vertex not covered by M. By Lemmas 4.2 and 4.3,

$$\left|\bigcup_{s,t} \mathcal{B}(X,Y,M,s,t)\right| \leq \sum_{s,t:\ s+t\geq 1} |\mathcal{B}(X,Y,M,s,t)| + \sum_{r\geq 1} |\mathcal{B}(X,Y,M,r)| \leq 2^{|X||Y|/2-n/300}.$$

Pick an arbitrary vertex $y \in Y$, define $X_0 = X \cup \{y\}, Y_0 = Y - \{y\}$ and $M_0 = M \cup \{xy\}$. Then by Lemma 4.5, we have

$$|\mathcal{G}(X_0,Y_0,M_0)| \geq (1+o(1))2^{|X_0||Y_0|/2} \geq 2^{|X||Y|/2-(|X|-|Y|)/2-1} \geq 2^{|X||Y|/2-2\beta n},$$

since $|X| - |Y| \leq 2\beta n$. Notice that any (X_0, Y_0, M_0) with $|X_0|$ even can be obtained from at most n different triples (X, Y, M) with $|X|$ odd in this way. Together with Lemma 4.4, it is sufficient to show that $\bigcup_{s,t} \mathcal{B}(X,Y,M,s,t)$ is negligible compared to $\mathcal{G}(X_0,Y_0,M_0)$:

$$\frac{\left|\bigcup_{s,t} \mathcal{B}(X,Y,M,s,t)\right|}{|\mathcal{G}(X_0,Y_0,M_0)|} \leq \frac{2^{|X||Y|/2-n/300}}{2^{|X||Y|/2-2\beta n}} \leq 2^{-n/400}.$$

This completes the proof of Theorem 1.1.

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