Ramsey numbers of cycles versus general graphs

John Haslegrave* Joseph Hyde† Jaehoon Kim ‡ Hong Liu†

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Abstract

The Ramsey number $R(F,H)$ is the minimum number N such that any N-vertex graph either contains a copy of F or its complement contains H. Burr in 1981 proved a pleasingly general result that for any graph H, provided n is sufficiently large, a natural lower bound construction gives the correct Ramsey number involving cycles: $R(C_n,H) = (n-1)(\chi(H)-1)+\sigma(H)$, where $\sigma(H)$ is the minimum possible size of a colour class in a $\chi(H)$-colouring of H. Allen, Brightwell and Skokan conjectured that the same should be true already when $n \geq |H|\chi(H)$.

We improve this 40-year-old result of Burr by giving quantitative bounds of the form $n \geq C|H|\log^4 \chi(H)$, which is optimal up to the logarithmic factor. In particular, this proves a strengthening of the Allen-Brightwell-Skokan conjecture for all graphs H with large chromatic number.

1 Introduction

For any pair of graphs F and H, Ramsey [26] famously proved in 1930 that there exists a number $R(F,H)$ such that given any graph G on at least $R(F,H)$ vertices either $F \subseteq G$ or $H \subseteq \overline{G}$. Determining $R(F,H)$ exactly for every pair of graphs F,H is a notoriously difficult problem in combinatorics. Indeed, $R(K_n,K_n)$ is not even known exactly for $n=5$. However, for some pairs of graphs, particularly when F and/or H are certain sparse graphs, $R(F,H)$ is known. We require some notation. For a graph H, define the chromatic number $\chi(H)$ of H to be the smallest number of colours in a proper colouring of H, that is, a colouring where no two adjacent vertices have the same colour. Further, let $\sigma(H)$ be the minimum possible size of a colour class in a $\chi(H)$-colouring of H.

Building on observations of Erdős [10] and Chvátal and Harary [8], Burr [4] constructed the following lower bound for $R(F,H)$ when F is a connected graph with $|F| \geq \sigma(H)$:

$$R(F,H) \geq (|F|-1)(\chi(H)-1)+\sigma(H). \tag{1}$$

The construction proving this bound is the graph G on $(|F|-1)(\chi(H)-1)+\sigma(H)-1$ vertices consisting of $\chi(H)-1$ disjoint cliques of size $|F|-1$ and an additional disjoint clique of size

*Mathematical Institute, Andrew Wiles Building, University of Oxford, Oxford OX2 6GG, UK (j.haslegrave@cantab.net). Supported by the UK Research and Innovation Future Leaders Fellowship MR/S016325/1.

†Mathematics Institute, Zeeman Building, University of Warwick, Coventry CV4 7AL, UK (Joseph.Hyde@warwick.ac.uk; h.liu.9@warwick.ac.uk). Supported by the UK Research and Innovation Future Leaders Fellowship MR/S016325/1.

‡Department of Mathematical Sciences, KAIST, South Korea (jaehoon.kim@kaist.ac.kr). Supported by the POSCO Science Fellowship of POSCO TJ Park Foundation and by the KAIX Challenge program of KAIST Advanced Institute for Science-X.

$\sigma(H)-1$. Clearly, since F is connected, $F \nsubseteq G$. Since $\overline{G}$ is the complete $\chi(H)$-partite graph with $\chi(H)-1$ vertex sets of size $|F|-1$ and one vertex set of size $\sigma(H)-1$, and the vertex set of size $\sigma(H)-1$ cannot completely contain any colour class of H, we have $H \nsubseteq \overline{G}$. In what follows we will often say a graph G is *H-free* to mean $H \nsubseteq G$.

Although the bound in (1) is very general, for some pairs of graphs it is extremely far from the truth. Indeed, Erdős [10] proved that $R(K_k, K_k) \geq \Omega(2^{k/2})$, whereas (1) only gives the much smaller lower bound of $(k-1)^2+1$. For some pairs of graphs however the bound in (1) is tight. For graphs F and H satisfying this lower bound $R(F,H) = (\chi(H)-1)(|F|-1)+\sigma(H)$, Burr and Erdős [5] coined the expression 'F is *H-good*'. The study of so-called *Ramsey goodness*, after being initiated by Burr and Erdős in 1983 [5], has attracted considerable interest. Prior to Burr's observation [4], already Erdős [10] had proved that the path on n vertices P_n was K_k-good; Gerenscér and Gyárfás [15] proved that for $n \geq k$ the path P_n is P_k-good; and Chvátal [7] proved every tree T is K_k-good. The result of Chvatal can be in fact viewed as a generalisation of Turán's theorem, which states that the complete balanced $(k-1)$-partite graph on N vertices, the so-called *Turán graph*, has the maximum number of edges amongst K_k-free graphs. Indeed, Turán's theorem is equivalent to the statement 'S_n is K_k-good', where S_n is the n-vertex star with $n-1$ leaves. To see this, let $N = (n-1)(k-1)$ and T be the complement of this Turán graph. Then T is precisely the construction in (1) when $(F,H) = (S_n, K_k)$. Moreover, T is the unique graph on $(n-1)(k-1)$ vertices without $S_n \subseteq T$ or $K_k \subseteq \overline{T}$. Add a vertex to T and denote the resulting graph by $T+v$. If v neighbours any vertex in T, then $S_n \subseteq T+v$. Otherwise, $K_k \subseteq \overline{T+v}$. That is, $R(S_n, K_k) = (n-1)(k-1)+\sigma(K_k) = (n-1)(k-1)+1$. This connection to Turán's theorem highlights how Ramsey goodness results can generalise other results in graph theory. See [13, 21, 22, 24, 25] and their references for more recent progress in the area of Ramsey goodness, as well as the survey of Conlon, Fox and Sudakov [9, Section 2.5].

In this paper we are specifically interested in when C_n, the n-vertex cycle, is H-good for general graphs H. This study can be traced back to Bondy and Erdős [3] who proved that C_n is K_k-good whenever $n \geq k^2-2$, which led Erdős, Faudree, Rousseau and Schelp [11] to conjecture that C_n is K_k-good whenever $n \geq k \geq 3$. Keevash, Long and Skokan [17] recently proved a strengthening of this conjecture for large k, showing that $n \geq C\frac{\log k}{\log\log k}$ suffices for some constant $C \geq 1$. For smaller k, Nikiforov [23] proved $n \geq 4k+2$ is sufficient and several authors have proved the conjecture for certain small values of k (see [6] and its references).

For Ramsey numbers of cycles versus general graphs H, Burr [4], in 1981, proved a satisfying result that C_n is H-good when n is sufficiently large as a function of $|H|$. It remains an intriguing open question to determine the threshold of n below which the Ramsey number $R(C_n, H)$ behaves differently from the natural construction of Burr yielding (1). In particular, Allen, Brightwell and Skokan [1] conjectured the following explicit bound.

Conjecture 1.1. *For any graph H and $n \geq \chi(H)|H|$, the cycle C_n is H-good, i.e. $R(C_n, H) = (\chi(H)-1)(n-1)+\sigma(H)$.*

Note that we may assume H to be a complete multipartite graph, as every H is a subgraph of some complete $\chi(H)$-partite graph H' with $\sigma(H') = \sigma(H)$, and clearly $R(C_n, H) \leq R(C_n, H')$.

Towards Conjecture 1.1, Pokrovskiy and Sudakov [25] very recently proved an important case when the graph H has polynomially small chromatic number: $|H| \geq \chi(H)^{23}$. More precisely, they showed that for $n \geq 10^{60}m_k$ and $m_1 \leq m_2 \leq \cdots \leq m_k$ satisfying $m_i \geq i^{22}$ for each i, $R(C_n, K_{m_1,\dots,m_k}) = (n-1)(k-1)+m_1$. Note, however, that it is well-known from random graph theory (e.g. [2]) that for almost all graphs H, the chromatic number is much larger: $\chi(H) = \Theta\big(\frac{|H|}{\log|H|}\big)$.

Our main result below is an almost optimal quantitative version of Burr's result [4]. In particular, this theorem holds for all graphs H. Moreover, as $x > C\log^4 x$ for all large x, it proves a strengthening of Conjecture 1.1 for all graphs H with large (constant) chromatic number.

Theorem 1.2. *There exists a constant $C > 0$ such that for any graph H and any $n \geq C|H| \log^4 \chi(H)$ we have that C_n is H-good, i.e. $R(C_n, H) = (\chi(H) - 1)(n - 1) + \sigma(H)$.*

The bound on n in Theorem 1.2 is best possible up to the logarithmic factor $\log^4 \chi(H)$. The following construction shows that the cycle length n has to be at least $(1 - o(1))|H|$ in order to be H-good.

Lower bound construction. Fix arbitrary $m, k \in \mathbb{N}$ with $k \geq 2$ and $0 < \varepsilon < \frac{1}{4}$. Set $n = (1-\varepsilon)mk$. Consider the complete k-partite graph H on partite set $V_1, \ldots, V_k$ with $|V_1| = (1 - \varepsilon)mk$ and $|V_2| = \cdots = |V_k| = \frac{\varepsilon mk}{k-1}$; so $|H| = mk$ and $\sigma(H) = \frac{\varepsilon mk}{k-1}$. Let G be the N-vertex graph, with $N = k(n - 1)$, consisting of k vertex disjoint cliques K_{n-1}. It is easy to see that G is C_n-free with H-free complement. Therefore, for these choices of C_n and H, we have

$$R(C_n, H) > k(n - 1) > (\chi(H) - 1)(n - 1) + \sigma(H).$$

Our proof takes a similar approach to the work of Pokrovskiy and Sudakov [25] and Keevash, Long and Skokan [17]. The bulk of the work is to prove a stability type result showing that C_n-free graphs G with H-free complement whose order is around the lower bound in (1) must be structurally close to Burr's construction. The two novel ingredients at the heart of our proof are (i) certain sublinear expansion properties and (ii) an 'adjuster' structure, both of which are inspired by recent work of Liu and Montgomery [20] on cycle embeddings in sublinear expanders. The theory of sublinear expanders has played a pivotal role in the resolutions of several old conjectures, see e.g. [14, 16, 18, 19, 20].

The logarithmic factor in our bound is an artifact of the use of sublinear expansion. Considering the above construction, it is not inconceivable that already when $n \geq (1 + o(1))|H|$, the cycle C_n is H-good. It would be interesting to at least get rid of the logarithmic factor and obtain a bound linear in $|H|$.

Organisation. The rest of the paper is organised as follows. We give an outline in Section 2. Preliminaries are given in Section 3. The two main ingredients, the sublinear expansion and the adjuster structure (and related lemmas) are given in Sections 4 and 5, respectively. The stability result and the proof of the main theorem are in Section 6.

2 Outline

Suppose that G has order at least $(\chi(H) - 1)(n - 1) + \sigma(H)$, but $\overline{G}$ is H-free. With the given condition on n, it is not too hard to find a cycle of length *at least* n in G; the difficulty lies in obtaining a cycle of the *precise* desired length n. A natural approach to deal with this is to create a sufficiently large structure which has inbuilt flexibility about the length of cycles it can produce. Pokrovskiy and Sudakov [25] use expansion properties to create some complex gadgets, which can be joined together to produce structures capable of producing paths of a wide range of lengths, up to almost all of their total size. However, the connectivity properties needed to link up these structures require n to be very large compared to $\chi(H)$, and in particular only produce a good bound on n if $|H|$ is at least a large power of $\chi(H)$.

To deal with the missing regime where $|H|$ is bounded by a polynomial in $\chi(H)$, we use an orthogonal approach. Instead of the complex gadgets featured in [25], we borrow ideas from recent work of Liu and Montgomery [20] to use certain sublinear expansion property (Definition 4.1) which allows the use of simpler gadgets we call adjusters (Definition 3.5). However, these adjusters are less flexible in terms of how much we can vary the length of the final cycle, relative to its total length. We circumvent this potential difficulty by creating a cycle which has enough adjusters to permit slightly more than $|H|$ different lengths, and also has a long adjuster-free section. The complement being H-free means that we may shorten this section until the cycle is close enough to the desired length, and use the flexibility from the adjusters to deliver the final blow.

One of the main technical difficulties in our proof is showing that a suitable expanding subgraph may be found in a graph with H-free complement under our definition of sublinear expansion. Additionally, we need to work directly with the specific graph H, rather than passing to a balanced multipartite graph, and the possibility that the parts of H are very unbalanced creates additional difficulties when one tries to use induction and it is necessary to reduce the number of parts first. We prove this, together with some useful consequences of expansion, in Section 4.

We then leverage these expansion properties to find the adjusters that we need in Section 5, and show how these may be combined together to give a suitable long cycle in a sufficiently well-connected subgraph. As a consequence of sublinear expansion and the fact that we do not need as much flexibility in length adjustment, we require only a very weak connectivity condition.

Finally, in Section 6 we establish a stability result on graphs G which have close to $(\chi(H)-1)(n-1)$ vertices, no copy of C_n, no copy of H in the complement, and which cannot be reduced to a smaller example by removing a part from H. That is, we show that a small number of vertices may be removed from such a graph G to leave $\chi(H)-1$ reasonably well-connected subgraphs of order close to n, whose complements exclude a complete bipartite graph H'; here our low connectivity requirements will obviate the need for a lower bound on the class sizes of H. The Ramsey goodness of the cycle then follows quickly from this stability result by considering how 2-connected blocks in G are linked.

3 Preliminaries

Our aim is to use induction on the number of partite sets k. We therefore define how we order possible graphs H. For a complete multipartite graph H, we write $H' \sqsubset H$ if there exists a graph H'' such that $H \cong H' \vee H''$, where $\vee$ denotes graph join, i.e. H is obtained by taking disjoint copies of H' and H'' and then adding all edges between $V(H')$ and $V(H'')$. Informally, H' consists of the subgraph induced by a proper subcollection of the parts of H. We will sometimes use $H' \sqsubseteq H$ to mean "$H' \sqsubset H$ or $H' = H$", i.e. H' is induced by a (not necessarily proper) subcollection of parts of H. For the main step of the induction, we will require both $H' \sqsubset H$ and $\sigma(H') = \sigma(H)$, but in some intermediate steps this latter condition is not needed. For a graph H with $\chi(H) = k$, we write $m = m(H) = \frac{|H|}{k}$ for the average part size of H, i.e. $|H| = km$.

For a graph G and vertex set $A \subset V(G)$, we define the *external neighbourhood* $N_G(A)$ to be the set $\{w \in V(G) \setminus A : vw \in E(G)\}$; note that this is disjoint from A. We omit the subscript if the graph is clear from context. The subgraph induced on A will be denoted by $G[A]$. We write $G - A = G[V(G) \setminus A]$ for the subgraph obtained by removing vertices in A. For disjoint subsets $A, B \subset V(G)$, an A-B path is a path between a vertex of A and a vertex of B. For two vertices $u, v \in V(G)$, the graph distance $\mathrm{dist}_G(u, v)$ is the length of a shortest u-v path in G.

Logarithms with no specified base are always taken to the base e throughout. For an integer t, we write $[t]$ for the set $\{1, 2, \dots, t\}$.

We will need the following result of Erdős and Szekeres [12].

Theorem 3.1. *Any sequence of at least $(r-1)^2+1$ integers contains a monotonic subsequence of length r.*

We use the following results of Pokrovskiy and Sudakov [25].

Corollary 3.2. *For $n \geq 10^{60}m_k$ and $m_1 \leq m_2 \leq \cdots \leq m_k$ satisfying $m_k \geq k^{22}$, we have $R(C_n, K_{m_1,\dots,m_k}) = (n-1)(k-1)+m_1$.*

Proof. Apply the aforementioned result [25] for $K_{m_1,\dots,m_k}$ with $m_i \geq i^{22}$ for each i to $K_{m'_1,\dots,m'_k}$ where $m'_1 = m_1$ and $m'_i = m_k$ if $i > 1$. For each $i > 1$ we have $m'_i = m_k \geq k^{22} \geq i^{22}$, and clearly $m'_1 \geq 1^{22}$. Thus if G is a C_n-free graph on $(n-1)(k-1)+m_1$ vertices, then $\overline{G}$ contains a copy of $K_{m'_1,\dots,m'_k}$, which in turn contains $K_{m_1,\dots,m_k}$. □

Our induction may reduce to the case when H is a complete bipartite graph.

Corollary 3.3 ([25, Corollary 3.8])**.** *Let n, m_1, m_2 be integers with $m_2 \geq m_1$, $m_2 \geq 8$, and $n \geq 2 \times 10^{49} m_2$. Then we have $R(C_n, K_{m_1,m_2}) = n + m_1 - 1$.*

The last one we need is an intermediate result on path embeddings.

Lemma 3.4 ([25, Lemma 3.7])**.** *Let n and m be integers with $n \geq 2 \times 10^{49} m$ and $m \geq 8$. Let G be a graph with $\overline{G}$ being $K_{m,m}$-free and $|N_G(A) \cup A| \geq n$ for every $A \subseteq V(G)$ with $|A| \geq m$. Let x and y be two vertices in G such that there exists an x-y path with order at least $8m$. Then there is an x-y path of order exactly n in G.*

3.1 Adjusters

In order to construct cycles covering a range of lengths, we will use the following graphs called *adjusters*. In the following definition, m and k are fixed numbers. When constructing adjusters, the relevant values of m and k will be clear from context.

Definition 3.5. For $r \geq 1$, an *r-adjuster* consists of r disjoint odd cycles $C_1, \dots, C_r$, with C_i having distinguished vertices v_i, w_i which are almost-antipodal (i.e. $\text{dist}_{C_i}(v_i, w_i) = (|C_i|-1)/2$), together with paths $P_1, \dots, P_r$ where the endpoints of P_i are w_i and v_{i+1} (subscripts taken modulo r), such that the paths are internally disjoint from each other and from the cycles, and $|C_i| \leq 2000 \log k \log(km)$. For the special case $r = 0$, a 0-adjuster is simply a cycle.

We refer to the cycles C_i as the *short cycles* of the adjuster. There are also cycles which use all the paths P_i and part of each short cycle; we refer to these as the *routes* of the adjuster. An r-adjuster contains 2^r routes, whose lengths are $r+1$ consecutive integers. We define the *length* of a adjuster to be the length of its longest route.

4 Sublinear expansions

Given a large graph whose complement is H-free we will pass to a subgraph which is not too large and has sublinear expansion properties, in which we can find an r-adjuster for some suitable r. After removing this r-adjuster we repeat the process with the remaining graph. We then join many adjusters together to create a very large adjuster with greater flexibility in the length of a route, and where this length can exceed n. We will shrink the structure, if necessary, to ensure that the maximum length of a cycle is not much more than n, without impairing this flexibility. We then show this structure contains C_n.

In this section we define the expansion properties we need, and show that any sufficiently large graph whose complement is F-free, for some graph F, contains an expanding subgraph of suitable size.

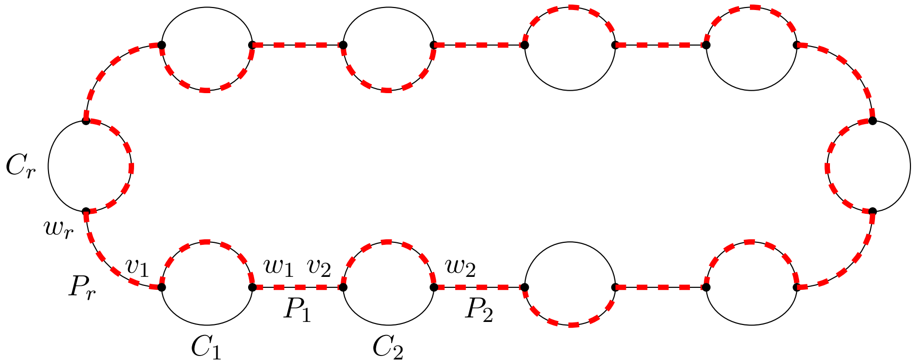

Figure 1: An r-adjuster. An example of a route in this r-adjuster is given by the red dashed line.

Definition 4.1. A graph G (Δ, β, d, k)-*expands into a set* $W \subseteq V(G)$ if the following holds.

- $|N_G(S) \cap W| \geq \Delta|S|$, for every $S \subseteq V(G)$ with $|S| \leq \beta d$.
- $|N_G(S)| \geq \frac{|S|}{10 \log k}$, for every $S \subseteq V(G)$ with $\beta d \leq |S| \leq |G|/2$.

Note that whenever we G say that (Δ, β, d, k)-expands, and do not specify the set G expands into, we mean that G (Δ, β, d, k)-expands into $W = V(G)$.

Lemma 4.2. *Fix a complete k-partite graph H of order mk, where $k \geq 2$, then for all $\beta, M, \Delta \geq 1$ with $M \geq 60\beta \geq 240\Delta$, $M \geq 10\beta\Delta$ and $M \geq 4k$, the following holds. Let G be a graph with $\overline{G}$ being H-free and $|G| \geq Mmk \log k$. Then there exists a subgraph $H' \sqsubseteq H$ induced by some collection of at least two parts and an induced subgraph $F \subseteq G$ such that the following holds:*

- *$\overline{F}$ is H'-free;*
- *$M|H'| \log \chi(H') - m(H') \leq |F| \leq M|H'| \log \chi(H')$.*
- *F is $(\Delta, \beta, m(H'), \chi(H'))$-expanding.*

Proof. We proceed by induction on k, considering two cases. The base case $k = 2$ will be covered by the inductive argument, since we argue that we may either (i) construct a suitable subgraph directly for $H' = H$, (ii) reduce to some smaller $H' \sqsubset H$ with at least two parts, or (iii) obtain a contradiction; in the base case only the first and third possibilities arise. Note that, by removing vertices if necessary, we may assume $|G| = Mmk \log k$ for $k \geq 2$. We write $m_1 \leq \cdots \leq m_k$ for the orders of the vertex classes of H.

The first case is if there exists a set $S \subset V(G)$ such that $m \leq |S| \leq |G|/2$ and $|N_G(S)| \leq |S|/(10 \log k) + m\Delta\beta$. Let $T = V(G) \setminus (N_G(S) \cup S)$. Since $M \geq 10\Delta\beta$, we have that

$$\begin{aligned} |T| \geq |G| - |N_G(S) \cup S| &\geq (1 - 1/(10 \log k))\,|G|/2 - m\Delta\beta \\ &= m\left(\tfrac{Mk}{2}(\log k - 1/10) - \Delta\beta\right) \\ &\geq km. \end{aligned}$$

We distinguish two subcases depending on whether $|S| \geq mk$ or $|S| < mk$.

First, suppose $|S| \geq mk$. We choose $t \in \mathbb{N}$ as large as possible such that $|T| \geq M|H_t| \log t$, where H_t is the graph induced by the t smallest parts of H. We then choose $s \in \mathbb{N}$ as large as possible so that $|S| \geq M|H_{s,t}| \log s$ where $H_{s,t}$ is the graph induced by the s smallest parts of H not contained in H_t; observe that $s, t \geq 1$ and $s, t \leq k - 1$.

We will quickly be able to resolve our first subcase after proving the following claim.

Claim 4.3. $s+t=k$.

Proof of claim. The claim is trivial for $k=2$, since $s,t\geq 1$. For all $k\geq 3$ we have

$$|T|\geq |G|/2-\frac{|G|}{20\log k}-\frac{|G|}{10k\log k}\geq M|H_{\lceil k/2\rceil}|\log(\lceil k/2\rceil),$$

which implies $t\geq \lceil k/2\rceil$; in particular, this proves the claim when $k=3$.

Assume for a contradiction that $s+t<k$, where $k\geq 4$ is fixed; as $t\geq \lceil k/2\rceil$, we have $s\leq (k-1)/2$. Maximalities of s and t ensure that

$$M|H_{s+1,t}|\log(s+1)>|S| \tag{2}$$

and

$$M|H_{t+1}|\log(t+1)>|T|. \tag{3}$$

Observe that H_{t+1} and $H_{s+1,t}$ both contain the $(t+1)$th smallest part of H.

Also, $|S|+|T|=|G|-|N_G(S)|>Mmk\log k-\frac{|S|}{10\log k}-m\Delta\beta$, hence by (2), (3) and the fact that $M\geq 10\beta\Delta$ we have

$$|H_{t+1}|\log(t+1)+|H_{s+1,t}|\log(s+1)c(k)\geq mk\log k-\frac{m}{10}, \tag{4}$$

where $c(k)=\left(1+\frac{1}{10\log k}\right)$. We want to show that (4) is in fact false, which provides the contradiction we need to prove Claim 4.3. To this end, we may assume $s+t=k-1$. It follows that

$$|H_{t+1}|+|H_{s+1,t}|=mk+m_{t+1}\leq mk+\frac{|H_{s+1,t}|}{s+1};$$

recall that m_{t+1} is the size of the $(t+1)$th smallest part of H. Consequently the LHS of (4) is at most

$$c(k)\log(s+1)|H_{s+1,t}|+\log(k-s)\left(mk-|H_{s+1,t}|\frac{s}{s+1}\right). \tag{5}$$

Note that $m(s+1)\leq |H_{s+1,t}|\leq mk$. For fixed s, (5) is linear in $H_{s+1,t}$, and consequently within this range it is maximised either at $|H_{s+1,t}|=m(s+1)$ or at $|H_{s+1,t}|=mk$.

We first consider the case $|H_{s+1,t}|=m(s+1)$, when (5) becomes

$$m(c(k)(s+1)\log(s+1)+(k-s)\log(k-s)).$$

This is a convex function of s, and so for $1\leq s\leq (k-1)/2$ is maximised when $s=1$ or when $s=(k-1)/2$. When $s=1$ we have

$$2c(k)\log 2+(k-1)\log(k-1)<k\log k-0.1 \tag{6}$$

for all $k\geq 4$. When $s=(k-1)/2$, we likewise have

$$(1+c(k))\frac{k+1}{2}\log\left(\frac{k+1}{2}\right)<k\log k-0.1 \tag{7}$$

for all $k\geq 4$.

Finally, we consider the case $|H_{s+1,t}|=mk$, when (5) becomes

$$mk(\log(s+1)c(k)+\log(k-s)/(s+1)).$$

Suppose $2 \le s \le \log(k-s)$. Then $\frac{s}{s+1}\log(k-s) \ge \frac{s^2}{s+1} \ge c(k)\log(s+1) + 0.1$, and so (5) is decreasing in $|H_{s+1,t}|$, and maximised when $|H_{s+1,t}| = m(s+1)$. Similarly if $s = 1$ and $k \ge 7$, (5) is decreasing. For $s = 1$ and $4 \le k \le 6$, by direct calculation (5) contradicts (4). Thus we may assume that $s \ge \log(k-s)$. Since $s \le (k-1)/2$ we have $s+1 \le k-s$. Thus

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\log(s+1)c(k) + \frac{\log(k-s)}{s+1}\right) = \frac{c(k)(s+1) - \frac{s+1}{k-s} - \log(k-s)}{(s+1)^2} > 0$$

and so $\log(s+1)c(k) + \log(k-s)/(s+1)$ is increasing in s for $\log(k-s) \le s \le (k-1)/2$. Hence within this range $mk(\log(s+1)c(k) + \log(k-s)/(s+1))$ is maximised when $s = (k-1)/2$, when we obtain

$$k\left(c(k) + \frac{2}{k+1}\right)\log\left(\frac{k+1}{2}\right) < k\log k - 0.1 \tag{8}$$

for all $k \ge 4$. Then (6), (7) and (8) together show that (5) is less than $mk\log k - \frac{m}{10}$, contradicting (4). ■

Hence $s + t = k$. Now either $\overline{G[T]}$ is H_t-free or $\overline{G[S]}$ is $H_{s,t}$-free, since otherwise disjointness of T and $N_G(S) \cup S$ ensures we have a copy of H in $\overline{G[S \cup T]}$. However, since $|S|, |T| \ge mk$, $\overline{G[T]}$ is not H_1-free and $\overline{G[S]}$ is not $H_{1,k-1}$-free. Thus in the former case we have $2 \le t \le k-1$ so we may apply the induction hypothesis to $G[T]$ with k replaced by t and H by H_t, and in the latter $2 \le s \le k-1$ so we may apply the induction hypothesis to $G[S]$ with k replaced by s and H by $H_{s,t}$. Here, such applications of the induction hypothesis are possible by the definition of s and t.

Secondly, suppose $m \le |S| < mk$. Note that, since $M \ge 10\Delta\beta$ and $M \ge 4k$, in this case

$$|T| \ge Mmk\log k - 2mk - m\Delta\beta \ge M(mk-1)\log(k-1).$$

Indeed, $Mmk\log\frac{k}{k-1} \ge Mm \ge 2mk + m\Delta\beta$ where we have used $\log(1+x) \ge \frac{x}{1+x}$ with $x = \frac{1}{k-1}$. Thus $|T| \ge M|H_{k-1,1}|\log(k-1)$ and $|S| \ge m \ge |H_1|$. Since H_1 is an independent set of size at most m, $\overline{G[S]}$ contains H_1. Thus $\overline{G[T]}$ must be $H_{k-1,1}$-free, as otherwise $\overline{G[S \cup T]}$ contains H, as before. Since $|T| \ge mk$, it is not $H_{1,1}$-free and so $k-1 \ge 2$. We may therefore apply the induction hypothesis to $G[T]$ with $k-1$ replacing k and $H_{k-1,1}$ replacing H.

The final case is where we have $|N_G(S) \cup S| \ge (1 + 1/(10\log k))\,|S| + m\Delta\beta$ for every set $S \subseteq V(G)$ with $m \le |S| \le |G|/2$. Consider the largest set $X \subseteq V(G)$ with $|X| \le 2m$ such that $|N_G(X)| \le \Delta|X|$. Since $\beta \ge 4\Delta$, we have that $|N_G(X) \cup X| \le (\Delta+1)|X| \le m\Delta\beta$. Thus, we must have $|X| < m$. Now consider $F = G - X$. If there is some $Y \subset V(F)$ with $|Y| \le m$ and $N_F(Y) < \Delta|Y|$, then

$$|N_G(X \cup Y) \cup (X \cup Y)| \le |N_G(X) \cup X| + |N_F(Y) \cup Y| < (\Delta+1)|X \cup Y|,$$

contradicting our choice of X.

For $|Y| \ge m$, we have

$$|N_F(Y) \cup Y| \ge |N_G(Y) \cup Y| - |X| \ge \left(1 + \frac{1}{10\log k}\right)|Y| + m(\Delta-1)\beta.$$

If $|Y| \le \beta m$ then this gives $|N_F(Y) \cup Y| \ge |Y| + (\Delta-1)|Y|$, whereas if $\beta m \le |Y| \le |F|/2$ then $|N_F(Y) \cup Y| \ge (1 + 1/10\log k)\,|Y|$. Thus F fulfils the requirements of Lemma 4.2 for $H' = H$. □

These expansion properties give us three abilities we will need to construct adjusters. The first is the ability to link together large sets, while avoiding a smaller set, via a path which is not too long. The main advantage of our stronger definition of expansion (compared to that in [25]) is that the length of path required is much shorter, being only of order $\log k \log|G|$, and the control we have over the size of the expanding subgraph means that we can ensure $\log|G| = O(\log k)$.

Lemma 4.4. *Suppose that G (Δ, β, d, k)-expands into a set $W \subseteq V(G)$, for $\Delta \geq 2$ and $k \geq 2$. If $A, B, C \subseteq V(G)$ are disjoint sets with $A, B \neq \varnothing$ such that $|C \cap W| \leq (\Delta - 2)\min\{|A|, |B|\}$ and $|C| \leq \beta d/(20 \log k)$, then there is an A-B path P in G avoiding C and with $|P| \leq 44 \log k \log|G|$.*

Proof. Set $A_0 = A$ and $A_{i+1} = (N(A_i) \cup A_i) \setminus C$ for each i. Since we have that $|A_{i+1}| \geq 2|A_i|$ for $|A_i| < \beta d$, and thus $|A_a| \geq \beta d$ where $a = \log_2(\beta d)$. For $i \geq a$ we have $|N(A_i)| \geq \frac{1}{10 \log k}|A_i| \geq 2|C|$, and so $|A_{i+1}| \geq (1 + 1/20 \log k)|A_i|$. Consequently $|A_b| \geq |G|/2$, where $b = a + \log_{1+1/20 \log k}(|G|/2\beta d)$. Applying $\log(1 + x) = -\log(1 - \frac{x}{1+x}) \geq (1 + 1/x)^{-1}$ for $x = 1/20 \log k$, and noting that $1 + 20 \log k \leq 22 \log k$, we get

$$b \leq a + 22 \log k \log|G| - 22 \log k \log(2\beta d) \leq 22 \log k \log|G|.$$

Since the same argument applies to B, we have that $A_b \cap B_b \neq \varnothing$, and so there is a path of length at most $2b \leq 44 \log k \log|G|$ avoiding C, as desired. □

The next ingredient follows from Lemma 4.4, and allows us to find a short odd cycle while retaining expansion into the rest of the graph.

Lemma 4.5. *Suppose G is not bipartite and (Δ, β, d, k)-expands into $W \subseteq V(G)$ for $\Delta \geq 4$, $k \geq 2$ and $\beta \geq 8\Delta$. Then G contains an odd cycle C with length at most $88 \log k \log|G|$, such that G also $(\Delta - 3, \beta, d, k)$-expands into $W \setminus V(C)$.*

Proof. Since G is not bipartite, it contains an odd cycle. Let C be a shortest odd cycle; it follows that if $x, y \in V(C)$ then $\mathrm{dist}_G(x, y) = \mathrm{dist}_C(x, y)$. Indeed, if not, choosing x, y to violate this condition with minimal distance, the shortest path between x and y is internally disjoint from the cycle, and adding this path to whichever path around the cycle has the appropriate parity gives a shorter odd cycle. Consequently $|C|$ is at most twice the diameter of G. Using Lemma 4.4 with $A_{4.4}, B_{4.4}$ being any singletons and $C_{4.4} = \varnothing$ implies the desired bound $|C| \leq 88 \log k \log|G|$.

Now, if C is a triangle then any vertex v can have at most 3 neighbours on C (trivially). Otherwise v can have at most two neighbours on C; indeed, if v has at least 3 neighbours on C then either v has two neighbours x, y with $\mathrm{dist}_C(x, y) \geq 3 > \mathrm{dist}_G(x, y)$, contradicting the earlier observation, or v has two neighbours x, y which are adjacent, giving a shorter odd cycle vxy. Thus for any set S we have $|N_G(S) \cap (W \setminus V(C))| \geq |N_G(S) \cap W| - 3|S|$, and, since G (Δ, β, d, k)-expands into W, G also $(\Delta - 3, \beta, d, k)$-expands into $W \setminus V(C)$. □

Finally, we show that we can construct two large, well-connected disjoint sets centred around any two given vertices.

Lemma 4.6. *Suppose G is a graph which (Δ, β, d, k)-expands into $W \subseteq V(G)$, and let x, y be distinct vertices. Then there exist disjoint sets $A, B \subset W \cup \{x, y\}$ of size $\beta d/2$ such that $x \in A$, and every vertex of A is connected to x by a path in A of length at most $\log_\Delta(\beta d/2)$, and similarly for B and y.*

Proof. We iteratively build up sets A_i, B_i with $|A_i| = |B_i| = \Delta^i$, such that any vertex in A_i is connected to x by a path in A_i of length at most i, until i is large enough that $|A_i| \geq \beta d/2$. We start from $A_0 = \{x\}, B_0 = \{y\}$. Given disjoint sets A_i, B_i with $|A_i| = |B_i| < \beta d/2$, we aim to choose disjoint sets $A' \subset N_G(A_i) \cap W$ and $B' \subset N_G(B_i) \cap W$ of size at least $\Delta|A_i|$. Then we can pick $A_{i+1} \subseteq A_i \cup (A' \setminus B_i)$ and $B_{i+1} \subseteq B_i \cup (B' \setminus A_i)$ with the required disjointness, connectivity and size properties.

First, take $A'' = (N(A_i) \cap W) \setminus N(B_i)$ and $B'' = (N(B_i) \cap W) \setminus N(A_i)$. If either of these, say A'' has size at least $\Delta|A_i|$, then we may choose $A' \subseteq A''$ and $B' \subseteq N(B_i) \cap W$ both of size $\Delta|A_i|$ (using the expansion property for the latter); by definition of A'' these sets are disjoint.

Consequently we may assume $|A''|, |B''| < \Delta|A_i|$. Since $|A_i \cup B_i| < \beta d$, the expansion property gives $|N(A_i) \cap N(B_i) \cap W| \geq 2\Delta|A_i| - |A''| - |B''|$, and so we can add disjoint subsets of $|N(A_i) \cap N(B_i) \cap W|$ to A'' and B'' to form sets A', B' as required. □

5 Constructing adjusters

In this section we prove the following result for some suitable constant C.

Lemma 5.1. *There exists a constant $C > 0$ satisfying the following. Fix integers m, k with $C \leq m \leq k^{22}$ and $n \geq 8 \times 10^{18} mk \log^4 k$. Let H be a complete k-partite graph on mk vertices, $H' \sqsubseteq H$ be $(s+1)$-partite for some $1 \leq s \leq k-1$ and m_2 be the size of the second smallest part of H. Let G be a graph of order at least $sn - n/10$ with $\overline{G}$ being H'-free, and such that any two sets of size at least $m_2 + 4.1 \times 10^{18} \log^4 k$ have at least $4.1 \times 10^{18} \log^4 k$ disjoint paths between them.*

- *If $s \geq 2$, then G contains a cycle of length exactly n.*
- *If $s = 1$ then G contains an r-adjuster for some $r \geq 9mk/8$ having length between $0.6n$ and $0.7n$.*

We first show how the expansion properties established in Section 4 can be used to construct adjusters.

Lemma 5.2. *Fix $k \geq 2$, $m \leq k^{22}$ and $n \geq 8 \times 10^5 mk \log^2 k$. Let H be a complete $(s+1)$-partite graph of order $m'(s+1)$, where $m' \in \mathbb{R}$, $1 \leq s \leq k-1$ and $|H| \leq mk$. Let G be a graph with $\overline{G}$ being H-free and $|G| \geq sn/10$. Then G contains an r-adjuster for some $r \geq \frac{mk}{2\times 10^4 \log^2 k}$, which has between mk and $mk + 2 \times 10^4 \log^2 k$ vertices.*

Proof. First note that we may assume, at the cost of replacing the upper bound with $|H| \leq 2mk$, that every part of H has order at least m'. Indeed, we may replace each part of H with order below m' with a part of order $\lceil m' \rceil$; this adds at most sm' vertices, and the new graph has average part size at most $2m'$. In particular, this means that if $H' \sqsubseteq H$ is a-partite of order ab then $b \geq m'$ and $ab \leq 2m'(s+1) \leq 2mk$. Clearly replacing H with a supergraph preserves the property that $\overline{G}$ is H-free.

Set

$$M = (10^4 k \log k) \times \frac{2m}{m'} \quad \text{and} \quad \beta = M/(200) \geq 100.$$

Then, as $1 \leq s \leq k-1$, we get

$$|G| \geq 8 \times 10^4 smk \log^2 k \geq M(2m'(s+1)) \log(s+1).$$

Further, note that $M \geq 4(s+1)$. We use Lemma 4.2 with $(M, \beta, \Delta, m, k)_{4.2} = (M, \beta, 10, m', s+1)$ to pass to a subgraph G' where $\overline{G'}$ is H'-free for some $H' \sqsubseteq H$ and which $(10, \beta, m'', s')$-expands into $V(G')$, where $|H'| = s'm''$ and $\chi(H') = s'$ with $2 \leq s' \leq s+1$, such that

$$Mm''s' \log s' - m'' \leq |G'| \leq Mm''s' \log s'. \tag{9}$$

In particular, since $m'' \geq m'$, we have that G' $(10, \beta, m', s')$-expands.

As $\overline{G'}$ is H'-free, G' has no independent set of order $|H'| = s'm''$, and it follows that $\chi(G') \geq |G'|/(s'm'') > 2$. Note that as $m \leq k^{22}$, from (9) we have that $|G'| \leq M(2m'(s+1)) \log k \leq 10^5 mk^2 \log^2 k \leq 10^5 k^{26}$. Thus, by Lemma 4.5 we may find an odd cycle C_1 of length at most $88 \log k \log|G'| \leq 10^4 \log^2 k$, such that we retain $(7, \beta, m', s')$-expansion into $V(G') \setminus V(C_1)$. Choosing almost-antipodal points v_1, w_1 on C_1 (i.e. $\mathrm{dist}_{C_1}(v_1, w_1) = (|C_1| - 1)/2$), we use Lemma 4.6 to find sets A_1 and B_1 of size $\beta m'/2$ with $A_1 \cap B_1 = \varnothing$, $A_1 \cap V(C_1) = \{v_1\}$ and $B_1 \cap V(C_1) = \{w_1\}$, with each vertex in A_1 (resp. B_1) having a path in A_1 to v_1 (resp. in B_1 to w_1) of length at most $\log_7(\beta m'/2) < 40 \log^2 k$.

Next we set $X = A_1 \cup V(C_1) \cup B_1$ and $Y = V(C_1)$. Note that, by (9) and that $|X| \leq \beta m' + 10^4 \log^2 k$, we have $|G' - X| \geq \frac{1}{2} M m'' s' \log s'$. We apply Lemma 4.2 with $(M, \beta, \Delta, m, k)_{4.2} = (\frac{M}{2}, \beta, 10, m'', s')$ again to find a subgraph G'' with $V(G'') \subset V(G') \setminus X$ which $(10, \beta, m''', s'')$-expands and has order at most $\frac{1}{2} M m''' s'' \log s''$, for some $2 \leq s'' \leq s'$ and $m''' \geq m'$. Within G'' we find another similar structure $A_2 \cup C_2 \cup B_2$, and since G' $(10, \beta, m', s')$-expands we can use Lemma 4.4, with parameters $(\Delta, \beta, d, k, A, B, C, W)_{4.4} = (10, \beta, m', s', A_2, B_1, V(C_1 \cup C_2), G')$ to find a path Q_1 between A_2 and B_1 that avoids $V(C_1 \cup C_2)$ and is of length at most $5000 \log^2 k$. We may assume Q_1 is disjoint from $A_1 \cup B_2$ since otherwise we may find a shorter path between (say) A_1 and A_2, then relabel appropriately. We then extend Q_1 by paths within A_2 to v_2 and within B_1 to w_1 to obtain a w_1-v_2 path P_1 of length at most $10^4 \log^2 k$.

Now we update X to $A_1 \cup V(C_1 \cup P_1 \cup C_2) \cup B_2$ (note that the unused parts of $A_2 \cup B_1$ are released) and Y to $V(C_1 \cup P_1 \cup C_2)$. As before, we choose G''' with $V(G''') \subset V(G') \setminus X$ such that G''' $(10, \beta, m^{(4)}, s''')$-expands for some $2 \leq s''' \leq s'$ and $m^{(4)} \geq m'$ and continue as before, updating X and Y. Observe that we can continue this process as long as $|G' - X| \geq \frac{1}{2} M m'' s' \log s'$. Thus, since $|G'| \geq M m'' s' \log s' - m''$, we can continue whenever $|X| \leq \frac{1}{4} M m'' s' \log s'$. Since $mk + \beta m' \leq \frac{1}{4} M m'' s' \log s'$, we can continue the process until $|Y| > mk$, as $|A_i|, |B_i| \leq \beta m'/2$ for all i. Moreover, at each stage of this process $|Y|$ increases by at most $2 \times 10^4 \log^2 k$. Thus as soon as $|Y| > mk$ we can stop the process and ensure $|Y| \leq mk + (2 \times 10^4) \log^2 k$ at this point. Since $mk + 2 \times 10^4 \log^2 k \leq \beta m'/20 \log(s+1) \leq \beta m'/20 \log s'$, we can then apply Lemma 4.4 with $(\Delta, \beta, d, k, A, B, C, W)_{4.4} = (10, \beta, m', s', A_1, B_r, Y, G')$ to find a path connecting the two remaining large sets A_1, B_r avoiding Y, which closes up the structure into an r-adjuster; since each cycle C_i and each path P_i built in this whole process has length at most $10^4 \log^2 k$ we have $r \geq \frac{mk}{2 \times 10^4 \log^2 k}$, and the adjuster contains all of Y, giving the required bounds. □

We will also need some long cycles to extend the adjusters we construct.

Lemma 5.3. *Fix $k \geq 2$, $m \leq k^{22}$ and $n \geq 8 \times 10^5 mk \log^2 k$. Let H be a complete $(s+1)$-partite graph of order $m'(s+1)$, where $m' \in \mathbb{R}$, $1 \leq s \leq k-1$ and $|H| \leq mk$. Let G be a graph with $\overline{G}$ being H-free and $|G| \geq sn/10$. Then G contains a cycle of length at least $n/((8 \times 10^5) \log^2 k)$.*

Proof. We proceed as in Lemma 5.2, with the following differences. First, set

$$M = \frac{n}{10m' \log(s+1)} \geq \frac{n}{10m' \log k} \quad \text{and} \quad \beta = M/200.$$

Secondly, at each stage rather than finding a cycle with two large sets, each of size $\beta m'/2$, attached we find a single edge with two large sets. This ensures that after each joining step we have a path with two large sets attached. Again, we continue until the total size of the path exceeds $\beta m'/40 \log k$, and then join the two large sets, using Lemma 4.4, to obtain a desired long cycle. □

In order to complete the proof of Lemma 5.1, we need to be able to join adjusters together while simultaneously retaining most of the flexibility of each adjuster and most of the length. This resembles Lemma 2.19 of [25], but the fact that in our case the length of an adjuster and the number of vertices are only loosely related creates extra difficulties.

Lemma 5.4. *Let F_i be an r_i-adjuster of length ℓ_i for $i \in \{1,2\}$, with F_1 and F_2 disjoint. Suppose there are $s > 16$ vertex-disjoint paths $P_1, \ldots, P_s$ between F_1 and F_2, each of length at most t. Then for some $i \neq j$ there is an r-adjuster of length ℓ contained in $F_1 \cup F_2 \cup P_i \cup P_j$, where*

$$(r_1 + r_2)(1 - 4s^{-1/2}) - 4 \leq r \leq r_1 + r_2$$

and

$$(\ell_1 + \ell_2)(1 - 4s^{-1/2}) \leq \ell \leq \ell_1 + \ell_2 + 2t.$$

Furthermore, if $r_2 = 0$ then the adjuster obtained contains a section of length $\ell - \ell_1$ with no short cycles.

Proof. We may assume each P_i is internally disjoint from $F_1 \cup F_2$, since otherwise we can replace it by a shorter path with this property. Write x_i for the endpoint of P_i in F_1, and y_i for the endpoint in F_2.

We will progressively reduce the number of paths to focus our attention on at least $s^{1/2}/4$ paths which relate to one another in a useful way. First, we give each path P_i an ordered pair of labels from $\{+, -\}$ as follows. If x_i lies on one of the r_1 short cycles of F_1, choose the first label to be $+$ if it is on the longer section of that cycle (that is, the longer section between the two vertices of that short cycle that have degree 3 in F_1), and $-$ if it is on the shorter section. If x_i lies on a path between cycles, choose the first label arbitrarily. Choose the second label similarly with respect to y_i. Now we may find a set of at least $s/4$ paths with identical label pairs. Note that this ensures there are routes C_1 around F_1 and C_2 around F_2 which cover all the x_i and y_i for P_i in this set of paths, with $|C_i| \in [\ell_i - r_i, \ell_i]$. In what follows we consider C_1 and C_2 to be oriented in a particular direction around F_1 and F_2, respectively. Renumbering, if necessary, we may assume these paths are $P_1, \ldots, P_{\lceil s/4 \rceil}$, and that $x_1, \ldots, x_{\lceil s/4 \rceil}$ are in order of their appearance in C_1. By the Erdős–Szekeres Theorem (Theorem 3.1), there is a subset of at least $t \geq \lceil \sqrt{s/4} \rceil \geq 2$ paths with endvertices y_i either in order of their appearance in C_2 or in reverse order. Again, by renumbering and reversing C_2 if necessary we may assume these are $P_1, \ldots, P_t$ and the y_i appear in reverse order.

If C_1 is the shortest route in F_1, let C_1' be the longest, and associate each vertex x_i for $i \in [t]$ with a vertex x_i', chosen as follows. If x_i is internal to a shorter side of a short cycle, let x_i' be the vertex on the longer side which is the same distance from the start vertex of the cycle (consequently, x_i' is one step further from the end of the cycle than x_i). If x_i is on one of the paths or is internal to the longer side of a short cycle, set $x_i' = x_i$. If C_2 is the shortest route in F_2, define C_2' and each y_i' similarly.

Now we show that some consecutive pair of paths work. Consider making a new adjuster from F_1, F_2, P_i and P_{i+1} for $i \in [t]$ (we take subscripts modulo t) as follows: starting from x_i, traverse P_i to y_i, then follow a route around F_2 to y_{i+1}, including both sections of any short cycle traversed in this route (apart from any short cycle containing y_i or y_{i+1}), traverse P_{i+1} to x_{i+1}, then follow a route around F_1 to x_i similarly. See Figure 2 for an example.

Write S_1 for the edges of C_1' between x_i' and x_{i+1}' and S_2 for the edges of C_2' between y_{i+1}' and y_i'. The short cycles of the new adjuster are those of the original two adjusters except for any which intersect S_1 or S_2. The longest route through the new adjuster uses all of C_1' except for at most $|S_1| + 1$ (the $+1$ added if $x_{i+1}' \neq x_{i+1}$), and all of C_2' except for at most $|S_2| + 1$ (the $+1$ added if $y_{i+1}' \neq y_{i+1}$). It also uses all of the two paths P_i and P_{i+1}, which contain at least one edge each, so its length is at least $\ell_1 + \ell_2 - |S_1| - |S_2|$. Note that the sets $S_1 \cup S_2$ obtained for different choices of $i \in [t]$ are disjoint.

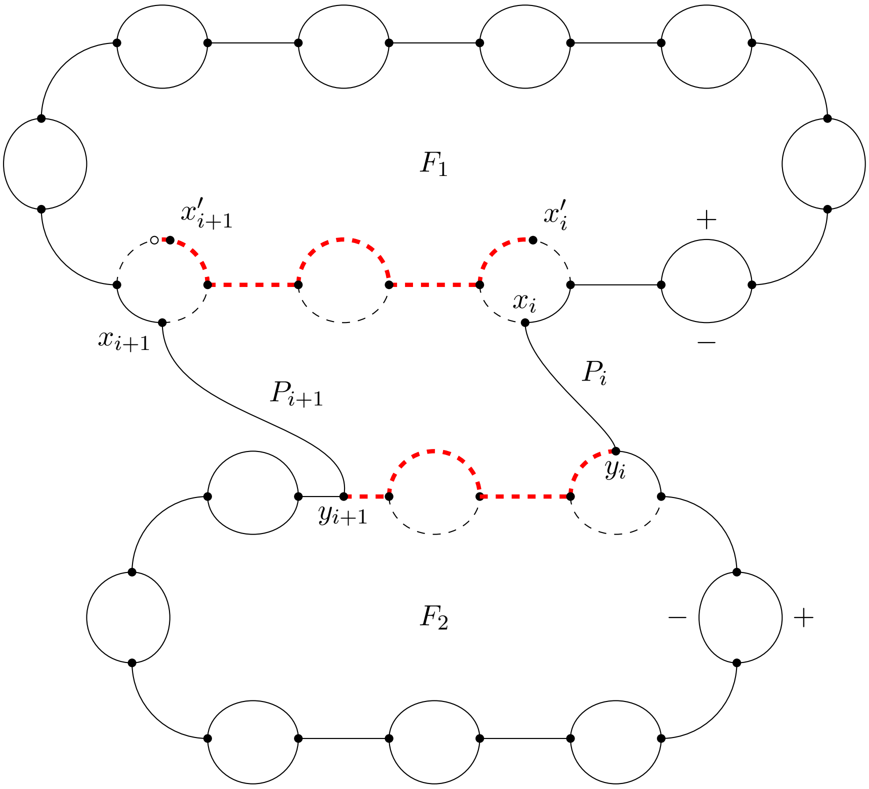

Figure 2: Creating a new adjuster using paths P_i and P_{i+1} with label $(-,+)$. Dashed sections are not part of the new adjuster, which has 5 fewer short cycles. The red section shows how much length is lost from F_1 and F_2; in the former case this includes one extra edge since x_{i+1} is on the short side of a cycle.

Now we claim that for some pair P_i, P_{i+1} the adjuster constructed has the required properties. Note that, since each path P_i has length at most t, the upper bound on length is satisfied for all adjusters constructed in this way, so it suffices to prove the lower bounds on ℓ and r. Suppose not, then for each i either more than $(r_1 + r_2)4s^{-1/2} + 4$ short cycles are lost, or $|S_1| + |S_2| > 4s^{-1/2}(\ell_1 + \ell_2)$. In the former case this means more than $(r_1 + r_2)4s^{-1/2}$ short cycles lie entirely in the section between x_i and x_{i+1} or the section between y_{i+1} and y_i, since only 4 can lie partially in these sections. Since these sections are disjoint, the former case occurs for fewer than $s^{1/2}/4$ choices of i. Similarly the latter case occurs for fewer than $s^{1/2}/4$ choices of i. Since there are at least $s^{1/2}/2$ choices for i, some choice gives an adjuster with the required properties.

Finally, note that the section of the adjuster obtained consisting of vertices not in F_1 is contiguous, and contributes at least $\ell - \ell_1$ to the overall length. If $r_2 = 0$ this section has no short cycles, as required. □

We are now ready to prove the main result of this section.

Proof of Lemma 5.1. Let $C = 10^{30}$. We will prove both parts of the statement simultaneously, by finding an r-adjuster of suitable length for some $r \geq 9mk/8$. If $s \geq 2$ this adjuster can be taken to have length slightly larger than n, such that one of the routes has length exactly n.

However, for $s = 1$ this is not possible, since G itself could have order less than n. In this case we show that the length of the adjuster can be made significantly larger than $n/2$.

We repeatedly apply Lemma 5.2, with $H_{5.2} = H'$ and $G_{5.2} = G - X_{i-1}$, where $X_0 = \varnothing$ and $X_i = V(F'_i)$ for $i \geq 1$ is the set of vertices used up from G (see the construction of F'_i, for each i, below), in order to find $f \leq 4 \times 10^4 \log^2 k$ adjusters F_i for $i \in [f]$, such that F_i is an r_i-adjuster and $r_{\text{total}} := \sum_{i=1}^{f} r_i \geq 2mk$. Note that, since $k^{22} \geq m \geq C = 10^{30}$ and $k \geq 10 \log^4 k$ for any $k \geq 2$, each F_i satisfies

$$|F_i| \geq mk \geq m_2 + 4.1 \times 10^{18} \log^4 k. \tag{10}$$

Also, the total size of all these adjusters (using the fact that every adjuster found has size at most $mk + 2 \times 10^4 \log^4 k \leq 1.1mk$) is at most $5 \times 10^4 mk \log^2 k$.

Set $F'_1 = F_1$ and $r'_1 = r_1$. After finding each F_i (for $i \geq 2$), we merge it with the adjuster F'_{i-1} using Lemma 5.4. There are at least $q = 4.1 \times 10^{18} \log^4 k$ vertex-disjoint paths between the two adjusters, by (10) and the hypothesis, and we may choose these to have length at most $|H'| + s \leq mk + k$ since the fact that $\overline{G}$ is H'-free implies that any longer path contains a short-cut. Thus we obtain a merged adjuster F'_i, which is an r'_i-adjuster with

$$(r'_{i-1} + r_i) \geq r'_i \geq (r'_{i-1} + r_i)(1 - 4q^{-1/2}) - 4, \tag{11}$$

and

$$|F'_i| \leq |F'_{i-1}| + |F_i| + 2mk + 2k. \tag{12}$$

It follows from (12) that for each $i \in [f]$,

$$|F'_i| \leq \sum_{j \leq i} |F_j| + (i-1)(2mk + 2k) \leq (5 \times 10^4) mk \log^2 k + 4 \times 10^4 \log^2 k (2mk + 2k) \leq n/10,$$

and so, setting $X_i = V(F'_i)$, we have $|G - X_{i-1}| \geq (sn - n/10) - n/10 \geq sn/10$. Consequently, we can indeed repeatedly apply Lemma 5.2 with $G_{5.2} = G - X_i$ for all $i \in [f]$.

Using (11), by induction we have $r'_{i-1} \leq \sum_{j<i} r_j$ for each $i \geq 2$, and so

$$r'_i \geq r'_{i-1} + r_i - 4q^{-1/2} r_{\text{total}} - 4. \tag{13}$$

Summing (13), we obtain $r'_f \geq r_{\text{total}}(1 - ((16 \times 10^4) q^{-1/2} \log^2 k) - 4f \geq 3r_{\text{total}}/4$.

We repeatedly apply Lemma 5.3 with $H_{5.3} = H'$ and $G_{5.3} = G - Y_j$, where $Y_1 = V(F'_f)$ and $Y_j = Y_1 \cup V(C'_j)$ for $j \geq 2$ is the set of vertices used up from G so far (see the construction of C'_i below), in order to find $c \leq 1.6 \times 10^6 \log^2 k$ cycles C_i each of length at least $n/(8 \times 10^5 \log^2 k) \geq m_2 + 4.1 \times 10^{18} \log^4 k$ (which we regard as 0-adjusters) satisfying the following bound on their total length:

- if $s \geq 2$ then $3n/2 \geq \sum_{i=1}^{c} |C_i| \geq 4n/3$; whereas
- if $s = 1$ then $3n/4 \geq \sum_{i=1}^{c} |C_i| \geq 2n/3$.

Set $C'_1 = C_1$. After finding each C_i (for $i \geq 2$), we merge it with the cycle C'_{i-1} using Lemma 5.4. As before, there are at least $q = 4.1 \times 10^{18} \log^4 k$ vertex-disjoint paths between the two cycles, by the hypothesis, and we may choose these to have length at most $|H'| + s \leq mk + k$. Thus we obtain a merged cycle C'_i with

$$(|C'_{i-1}| + |C_i|)(1 - 4q^{-1/2}) \leq |C'_i| \leq |C'_{i-1}| + |C_i| + 2mk + 2k. \tag{14}$$

By induction, and that after each merging of cycles we can truncate the length of the new cycle (just as we did with the paths), we have $|C'_{i-1}| \leq \left(\sum_{j<i}|C_j|\right) + 2mk + 2m$ for each $i \geq 2$, and so

$$|C'_i| \geq |C'_{i-1}| + |C_i| - 4q^{-1/2}\left(\left(\sum_{j=1}^{c}|C_j|\right) + 2mk + 2m\right). \qquad (15)$$

Summing (15), we obtain

$$|C'_c| \geq \sum_{j=1}^{c}|C_j|(1 - (6.4 \times 10^6)q^{-1/2}\log^2 k) - ((6.4 \times 10^6)q^{-1/2}\log^2 k)(2mk + 2m),$$

hence for $s \geq 2$, $|C'_c| \geq 5n/4$ and for $s = 1$, $|C'_c| \geq 0.62n$.

Now merge C'_c with F'_f using Lemma 5.4 to produce our desired r-adjuster. One can see that $r \geq (3/4)^2 r_{\text{total}} \geq 9mk/8 \geq (m+1)k$.

Suppose $s \geq 2$. Note that the final use of Lemma 5.4 merges a adjuster of length less than n with a 0-adjuster of length at least $5n/4$, and thus our r-adjuster of length $\ell > n$ contains a section of length at least $\ell - n$ in which there are no short cycles. Using the fact that $\overline{G}$ is H'-free we can shortcut parts of this section, if necessary, without reducing r, until $0 \leq \ell - n \leq mk + k$. Now, using the fact that it is an r-adjuster for some $r \geq (m+1)k$, we may find a route of length exactly n.

Finally, if $s = 1$ the final adjuster similarly has length $\ell \geq 0.6n$ and contains a section of length $\ell - 0.6n$ with no short cycles. We can shorten this section, if necessary, as before to obtain an r-adjuster of length between $0.6n$ and $0.7n$. □

6 Stability and exactness

In this section we prove our main result via the following statement.

Theorem 6.1. *There exists a constant $C' > 0$ such that for any complete k-partite graph H on mk vertices where $C' \leq m \leq k^{22}$ and $n \geq 10^{60} mk \log^4 k$, we have that C_n is H-good, i.e. $R(C_n, H) = (\chi(H) - 1)(n - 1) + \sigma(H)$.*

Proof of Theorem 1.2. Set $k = \chi(H)$ and $m = |H|/k$. By adding edges to H, if necessary, without changing $\sigma(H)$, we may assume it is a complete k-partite graph for $k = \chi(H)$. If $m > k^{22}$ then the same is true of the size of the largest part, and so Corollary 3.2 gives the required bound for some constant C'' (and in fact the logarithmic term is not needed). If $C' \leq m \leq k^{22}$ then Theorem 6.1 gives the required bound with constant 10^{60}. Finally, if $m < C'$ then by increasing the size of the largest part we may replace H with a graph H' of order at most $C'|H|$ and $\sigma(H') = \sigma(H)$ which satisfies the conditions of Theorem 6.1, giving that C_n is H'-good (and hence H-good) for $n \geq 10^{60} C'|H| \log^4 \chi(H)$. □

To prove Theorem 6.1, we first establish the following stability result.

Lemma 6.2. *There exists a constant $C > 0$ satisfying the following. Fix $k \geq 2$ and $C \leq m \leq k^{22}$ and $z \geq 0$ and $n \geq 10^{49} km \log^4 k$. Let H be a complete k-partite graph with mk vertices, and write $m_1 \leq \cdots \leq m_k$ for the sizes of the parts. Define $\hat{H} \sqsubset H$ to be the graph obtained by removing a part of size m_2. Suppose G is a C_n-free graph with $|G| \geq (k-1)(n-1) + z$ such that $\overline{G}$ is H-free. Then at least one of the following holds.*

(i) There exists $G' \subset G$ such that $\overline{G'}$ is $\hat{H}$-free and $|G'| \geq (k-2)(n-1) + z$.

(ii) There exist disjoint sets $A_1, \dots, A_{k-1}$, with no edges between them, such that for each i we have

- $|A_i| \geq 0.95n$,
- $\overline{G[A_i]}$ *is* K_{m_1,m_2}*-free, and*
- *within* $G[A_i]$*, any two disjoint sets of size at least* $m_2 + 4.1 \times 10^{18} \log^4 k$ *have at least* $4.1 \times 10^{18} \log^4 k$ *disjoint paths between them.*

Proof. Let C be a constant given in Lemma 5.1. Suppose not. Start with $A = V(G)$ and $S = \varnothing$. If there exists a set X of order at most $4.1 \times 10^{18} \log^4 k$ such that removing X from $G[A]$ increases the number of components of $G[A]$ of order at least m_2, remove X from A and add it to S. Do this for k iterations or until no such set exists. At the end of the process we have $|S| \leq 4.1 \times 10^{18} k \log^4 k$.

Note that any set B of order at least m_2 satisfies $|B \cup N(B)| \geq n$, else $G[V(G) \setminus (B \cup N(B))]$ satisfies (i). In particular, if $B \subset A$ with $|B| \geq m_2$ then

$$|B \cup N_{G[A]}(B)| \geq |B \cup N(B)| - |S| \geq n - 4.1 \times 10^{18} k \log^4 k \geq |H|.$$

It follows that the total size of all components of $G[A]$ of size less than m_2 is less than m_2, since otherwise there is a union of some of these components of size between m_2 and $2m_2$, contradicting the fact that sets of this size expand well. It also follows that any other component has order at least $n - 4.1 \times 10^{18} k \log^4 k$; let these components of $G[A]$ be $A_1, \dots A_t$, where $t \geq 1$ and

$$\sum_{i \in [t]} |A_i| \geq |A| - m_2 \geq |G| - |S| - m_2 \geq (k-1)n - k - 4.1 \times 10^{18} k \log^4 k - m_2. \tag{16}$$

We must have $t \leq k-1$ since otherwise $\overline{G}$ contains a copy of H obtained by choosing a suitable number of vertices from $A_1, \dots, A_k$.

In particular, we added vertices to S fewer than k times, and so the process described above stopped because no more removals were possible. This will give the required connectivity properties. Indeed, if B_1, B_2 are disjoint sets in A_i of size at least $m_2 + 4.1 \times 10^{18} \log^4 k$ then no set X of order $4.1 \times 10^{18} \log^4 k$ separates $B_1 \setminus X$ from $B_2 \setminus X$ within $G[A_i]$ (since both these sets have size at least m_2 and by the construction of A). Thus, by Menger's theorem there are at least $4.1 \times 10^{18} \log^4 k$ disjoint paths between them in $G[A_i]$.

For each $i \leq t$, we define $k-t+1$ subgraphs $H_i^{(1)} \sqsubset \cdots \sqsubset H_i^{(k-t+1)} \sqsubset H$ as follows. Set $H_i^{(1)}$ to a set of m_{k+1-i} vertices, and for $2 \leq j \leq k-t+1$ set $H_i^{(j)} = K_{m_{k+1-i}, m_{k-t}, m_{k-t-1}, \dots, m_{k-t-j+2}}$, i.e. $H_i^{(j)}$ is a complete j-partite subgraph consisting of the ith largest class of H together with $j-1$ of the $k-t$ smallest classes taken in decreasing order.

For each $i \leq t$, let $s_i \geq 1$ be the largest value such that $\overline{G[A_i]}$ contains $H_i^{(s_i)}$. Suppose that $\sum_{i \in [t]} s_i \geq k$, and consider the graph induced in $\overline{G}$ by the vertices of a copy of $H_i^{(s_i)}$ in $\overline{G[A_i]}$ for each i. Taking the largest class from each copy creates a copy of $K_{m_k, \dots, m_{k-t+1}}$, and the other classes give a complete $(k-t)$-partite graph with jth largest part having size at least $m_{k-t+1-j}$ for each $j \in [k-t]$, which contains $K_{m_{k-t}, \dots, m_1}$. Thus $\overline{G} \supset H$, a contradiction.

Consequently we have $\sum_{i \in [t]} s_i \leq k-1$, and in particular $s_i \leq k-t$ for each i. By definition of s_i, $\overline{G[A_i]}$ is $H_i^{(s_i+1)}$-free for each i. If $s_i > 1$ we have $|A_i| < s_i n - n/10$, since otherwise Lemma 5.1, taking $H'_{5.1} = H_i^{(s_i+1)}$, gives a copy of C_n inside $G[A_i]$. Contrariwise, if $s_i = 1$ we have $|A_i| < n + m_{k-t}$ by Corollary 3.3. In either case, for each $i \in [t]$ we have $|A_i| < s_i n + m_{k-t}$.

Note that $(t-1)m_{k-t} \le |H| = mk$. Thus, if for any i we have $s_i > 1$ then

$$\sum_{j\in[t]} |A_j| < s_i n - n/10 + \sum_{\substack{j\in[t]\\ j\neq i}} (s_i n + m_k - t) \le (k-1)n - n/10 + (t-1)m_{k-t},$$

contradicting (16). We may therefore assume each $s_i = 1$. Similarly if $\sum_{i\in[t]} s_i < k-1$, and so $t < k-1$, we obtain a contradiction to (16). Thus $t = k-1$, and each of $A_1, \ldots, A_{k-1}$ has size at least $n - 4.1 \times 10^{18} k \log^4 k \ge 0.95n$. If for some i we have $K_{m_1,m_2} \subset \overline{G[A_i]}$, then taking the copy of K_{m_1,m_2} together with m_k vertices from each other part gives a copy of $K_{m_k,\ldots,m_k,m_2,m_1}$ (with k parts in total) in $\overline{G}$, and so $\overline{G} \supset H$, a contradiction. Together with the connectivity properties heretofore established, this completes the proof. □

We now proceed from the stability result above to exactness. We need the following fact that when the complement is K_{m_1,m_2}-free we can do much better than Lemma 5.4.

Lemma 6.3. *Let F_i be an r_i-adjuster of length ℓ_i for $i \in \{1,2\}$, with F_1 and F_2 disjoint, and suppose $\overline{G[V(F_i)]}$ is K_{m_1,m_2}-free for each i, where $m_2 \ge m_1$. Suppose further that there are two vertex-disjoint paths P_1, P_2 between F_1 and F_2, each of length at most t. Then there is an r-adjuster of length ℓ all of whose vertices are contained in $F_1 \cup F_2 \cup P_1 \cup P_2$, where $r \ge r_1 + r_2 - 2(m_1+m_2)$ and $\ell_1 + \ell_2 - 2(m_1+m_2) \le \ell \le \ell_1 + \ell_2 + 2t$. Furthermore, if $r_2 = 0$ then the adjuster contains a section of length at least $\ell - \ell_1$ with no short cycles.*

Proof. Let P_i have ends x_i in F_1 and y_i in F_2. We claim that there is a path Q_1 of length at least $\ell_1 - m_1 - m_2 - 1$ in F_1 between x_1 and x_2. To see this, first note that unless x_1 and x_2 are on opposite sides of the same short cycle, there is a route in F_1 which includes both x_1 and x_2, and has length at least $\ell_1 - 2$. If x_1 and x_2 are at distance at most m_2 in this route then by taking the whole route except for a path of length at most m_2 between the two vertices, we are done. If not, then the m_1 vertices immediately before x_1 on the route do not include x_2, and the m_2 vertices immediately before x_2 do not include x_1. There is an edge $z_1 z_2$ between these two sets of vertices in $G[V(F_1)]$, since $\overline{G[V(F_1)]}$ is K_{m_1,m_2}-free, and so there is a path which starts at x_1, goes around the route to z_2, via the extra edge to z_1 and then around the route in the opposite direction to x_2. This path misses out at most $m_1 + m_2 - 1$ vertices from the route, so has at least the required length. See Figure 3. Finally, if x_1 and x_2 are on opposite sides of the same short cycle, there is a path around the adjuster from x_1 to x_2 using the the vertices on that short cycle after x_1 and before x_2, and the longer side of every other short cycle. Similarly there is another path which uses the vertices on that short cycle after x_2 and before x_1. At least one of these uses at least half of every short cycle, so has length at least ℓ_1.

Now taking this path together with a corresponding path for F_2 and the two paths P_1, P_2 between them gives a cycle C of length between $\ell_1 + \ell_2 - 2(m_1+m_2)$ and $\ell_1 + \ell_2 + 2t$. We extend this to an adjuster by including all short cycles from F_1 and F_2 such that one of the routes round that short cycle is entirely contained in C. Since all but at most $m_1 + m_2$ edges of a route of F_i are included in the cycle, this means at most $2(m_1+m_2)$ short cycles of F_1 or F_2 are not included, giving $r \ge r_1 + r_2 - 2(m_1+m_2)$. Finally, note that the section of the adjuster obtained consisting of vertices not in F_1 is contiguous, and contributes at least $\ell - \ell_1$ to the overall length. If $r_2 = 0$ this section has no short cycles, as required. □

This enables us to effectively deal with the case where two of the A_i (or, more precisely, the expanding subgraphs within them) are connected by two disjoint paths.

Proof of Theorem 6.1. It suffices to prove, via induction on k, the conclusion for $2C\frac{k-1}{k} \le m \le k^{22}$, where C is at least the constants in Lemma 5.1 and Lemma 6.2. Since we have $C \le 2C\frac{k-1}{k} \le 2C$, we can then take $C' = 2C$ to conclude.

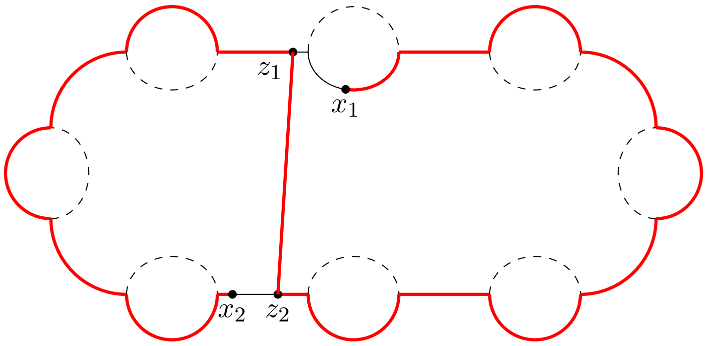

Figure 3: A long x_1-x_2 path using the extra edge z_1z_2 (shown in bold, and highlighted if colour is available). Dashed lines indicate parts of the adjuster not in the relevant route.

Suppose it is not true, and take a counterexample, i.e. a graph G of order $(k-1)(n-1)+\sigma(H)$ that is C_n-free and with H-free complement. We apply Lemma 6.2. If (i) applies then we have a graph $G' \subset G$ of order $(k-2)(n-1)+\sigma(H)$, with $\hat{H}$-free complement. Note that by definition of $\hat{H}$ we have $\sigma(H) = \sigma(\hat{H})$. Also, since $m_2 \leq \frac{mk}{k-1}$, we have

$$m(\hat{H}) \geq m\frac{k(k-2)}{(k-1)^2} \geq 2C\frac{k-2}{k-1}.$$

Thus if $|\hat{H}| \leq (k-1)^{23}$ we have $G' \supset C_n$ by the induction hypothesis, whereas otherwise we have $G' \supset C_n$ by Corollary 3.2. Consequently we must have (ii).

Within each set A_i we apply Lemma 4.2 with $G_{4.2} = G[A_i]$, $H_{4.2} = K_{m_1,m_2}$, $M_{4.2} = |A_i|/((m_1+m_2)\log 2)$ and $\beta_{4.2} = M_{4.2}/100 > 40$ to find a $(10, M_{4.2}/100, (m_1+m_2)/2, 2)$-expander H_i of order at least $|A_i| - (m_1+m_2)/2 \geq |A_i| - m$.

Suppose that H_i and H_j are linked (in G) by two disjoint paths for some distinct $i, j \in [k-1]$. Recall that $|A_i| \geq 0.95n$. Applying Lemma 5.1 to $G[A_i]$, we find an r-adjuster F_i for $r \geq 9mk/8$ with length between $0.6n$ and $0.7n$. Similarly, we may find an adjuster in A_j of length between $0.6n$ and $0.7n$; however, instead of using it as an adjuster we simply take the longest route to find a cycle C_j of that length.

Note that the size of the adjuster and cycle ensures that they intersect H_i and H_j respectively in at least $|H_i| + 0.6n - |A_i| \geq 0.5n$ vertices. Suppose that the disjoint paths between H_i and H_j have endpoints x_1 and x_2 in H_i. By Lemma 4.6, we may find disjoint sets $B_1 \ni x_1$ and $B_2 \ni x_2$ in $V(H_i)$ of size $|A_i|/200$, such that every vertex in B_1 is connected to x_1 by a path, and likewise for B_2 and x_2. We may therefore use Lemma 4.4 with $A_{4.4} = V(F_i) \cap V(H_i), B_{4.4} = B_1 \cup B_2, C_{4.4} = \varnothing$ to give a path from $V(F_i)$ to B_1 or B_2, without loss of generality the former. Extend it to a path from $V(F_1)$ to x_1 using vertices from B_1, and remove vertices from this path, if necessary, to obtain a shortest path P_1. This ensures that $|V(P_1)| \leq m_1 + m_2 + 1$ since $\overline{G[A_i]}$ is K_{m_1,m_2}-free; note also that P_1 is disjoint from B_2. Next apply Lemma 4.4 with $A_{4.4} = V(F_i) \cap V(H_i) \setminus V(P_1), B_{4.4} = B_2, C_{4.4} = V(P_1)$ to give a disjoint x_2-$V(F_i)$ path P_2. We can similarly find two disjoint paths to C_j in H_j, and the union of these with the existing paths between H_i and H_j give two disjoint paths between the adjuster and the cycle. Since $\overline{G}$ is H-free, each of these paths contains a shortest path of length at most $mk+k$. We apply Lemma 6.3 to merge the adjuster and cycle using these shortest paths, obtaining an adjuster with at least $9mk/8 - 2(m_1+m_2) \geq mk+k$ short cycles, of total length ℓ between $1.2n - 2(m_1+m_2) > 1.19n$ and $1.4n + 2(mk+k) < 1.41n$. Recall that F_i has length at most $0.7n$, so the adjuster obtained from merging F_i and C_j contains a section of length at least $\ell - 0.7n \geq 0.49n > \ell - n$ without short cycles (the part corresponding to C_j). We may reduce this section until the total length is between n and $n+mk+k$, since $\overline{G}$ is H-free, and then use the reducing power of the adjuster to obtain a cycle of length exactly n.

Thus we may assume that no pair H_i, H_j is linked by two disjoint paths. It follows that for some index i there is a vertex v which separates H_i from all other H_j. Let A be the component of $G-v$ containing $H_i - v$ and let $G' = G[A \cup \{v\}]$. Note that $\overline{G'}$ is K_{m_1+1,m_2+1}-free, since otherwise we may choose m_k vertices from $H_j - v$ for each $j \neq i$ together with the vertices of a K_{m_1,m_2} in $\overline{G'-v}$ to give a graph containing H in $\overline{G}$. Recall that (i) of Lemma 6.2 does not apply. It follows that any set $B \subset A$ of size m_2 has $|N_G(B) \cup B| \geq n$, since otherwise $\overline{G-(N_G(B)\cup B)}$ is not $\hat{H}$-free, implying that $\overline{G}$ is not H-free. Since also $N_{G'}(B) \cup B = N_G(B) \cup B$, we have $|N_{G'}(B) \cup B| \geq n$. Thus, for any set $B' \subset V(G')$ of order at least m_2+1, we have $|N_{G'}(B') \cup B'| \geq n$ (since B' contains a set of order m_2 not including v). Now, using Lemma 5.3 with $G_{5.3} = H_i$ and $H_{5.3} = K_{m_2,m_2}$, we may find a cycle of length at least $10m_2$ in H_i. Taking x, y to be two consecutive vertices on this cycle, Lemma 3.4 applies to $G_{3.4} = G'$ with $m_{3.4} = m_2 + 1$, and we may find a path of order exactly n between x and y, giving a copy of C_n. □

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